UNITAT 12

CÀLCUL DE PRIMITIVES

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Concepte de primitiva

NOMBRES I POTÈNCIES SENZILLS

1. a) $\int 1\,dx = x$ b) $\int 2\,dx = 2x$ c) $\int \sqrt{2}\,dx = \sqrt{2}\,x$

2. a) $\int 2x\,dx = x^2$ b) $\int x\,dx = \frac{x^2}{2}$ c) $\int 3x\,dx = \frac{3x^2}{2}$

3. a) $\int 7x\,dx = \frac{7x^2}{2}$ b) $\int \frac{x}{3}\,dx = \frac{x^2}{6}$ c) $\int \sqrt{2}\,x\,dx = \frac{\sqrt{2}\,x^2}{2}$

4. a) $\int 3x^2\,dx = x^3$ b) $\int x^2\,dx = \frac{x^3}{3}$ c) $\int 2x^2\,dx = \frac{2x^3}{3}$

5. a) $\int 6x^5\,dx = x^6$ b) $\int x^5\,dx = \frac{x^6}{6}$ c) $\int 3x^5\,dx = \frac{3x^6}{6} = \frac{x^6}{2}$

POTÈNCIES D'EXPONENT ENTER

6. a) $\int (-1)x^{-2}\,dx = x^{-1} = \frac{1}{x}$ b) $\int x^{-2}\,dx = \frac{x^{-1}}{-1} = \frac{-1}{x}$ c) $\int \frac{5}{x^2}\,dx = \frac{-5}{x}$

7. a) $\int \frac{2}{x^3}\,dx = \frac{-1}{x^2}$ b) $\int \frac{1}{x^3}\,dx = \frac{-1}{2x^2}$

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8. a) $\int \frac{1}{(x-3)^5}\,dx = \frac{-1}{4\,(x-3)^4}$ b) $\int \frac{5}{(x-3)^3}\,dx = \frac{-5}{2\,(x-3)^2}$

LES ARRELS TAMBÉ SÓN POTÈNCIES

9. a) $\int \frac{3}{2}x^{1/2}\,dx = x^{3/2} = \sqrt{x^3}$ b) $\int \frac{3}{2}\sqrt{x}\,dx = \int \frac{3}{2}x^{1/2} = x^{3/2} = \sqrt{x^3}$

10. a) $\int \sqrt{x}\,dx = \frac{2}{3}\int \frac{3}{2}x^{1/2} = \frac{2}{3}x^{3/2} = \frac{2}{3}\sqrt{x^3}$ b) $\int 7\sqrt{x}\,dx = 7\int \sqrt{x} = \frac{14}{3}\sqrt{x^3}$

11. a) $\int \sqrt{3x}\, dx = \int \sqrt{3}\sqrt{x} = \sqrt{3}\int \sqrt{x} = \frac{2\sqrt{3}}{3}\sqrt{x^3} = \frac{2\sqrt{3x^3}}{3}$

b) $\int \frac{\sqrt{2x}}{5}\, dx = \int \frac{\sqrt{2}}{5}\sqrt{x} = \frac{\sqrt{2}}{5}\int \sqrt{x} = \frac{\sqrt{2}}{5} \cdot \frac{2}{3}\sqrt{x^3} = \frac{2\sqrt{2}}{15}\sqrt{x^3} = \frac{2\sqrt{2x^3}}{15}$

12. a) $\int \frac{1}{2} x^{-1/2}\, dx = x^{1/2} = \sqrt{x}$

b) $\int \frac{1}{2\sqrt{x}}\, dx = \sqrt{x}$

13. a) $\int \frac{3}{2\sqrt{x}}\, dx = 3\int \frac{1}{2\sqrt{x}} = 3\sqrt{x}$

b) $\int \frac{3}{\sqrt{5x}}\, dx = \frac{6}{5}\int \frac{5}{2\sqrt{5x}} = \frac{6}{5}\sqrt{5x}$

14. a) $\int \sqrt{x^3}\, dx = \int x^{3/2} = \frac{x^{5/2}}{5/2} = \frac{2}{5}\sqrt{x^5}$

b) $\int \sqrt{7x^3}\, dx = \sqrt{7}\int \sqrt{x^3} = \frac{2}{5}\sqrt{7x^5}$

15. a) $\int \frac{1}{x}\, dx = \ln |x|$

b) $\int \frac{1}{5x}\, dx = \frac{1}{5}\int \frac{5}{5x} = \frac{1}{5} \ln |5x|$

16. a) $\int \frac{1}{x+5}\, dx = \ln |x+5|$

b) $\int \frac{3}{2x+6} dx = \frac{3}{2}\int \frac{2}{2x+6} = \frac{3}{2} \ln |2x+6|$

ALGUNES FUNCIONS TRIGONOMÈTRIQUES

17. a) $\int \cos x\, dx = \sin x$

b) $\int 2\cos x\, dx = 2 \sin x$

18. a) $\int \cos\left(x + \frac{\pi}{2}\right) dx = \sin\left(x + \frac{\pi}{2}\right)$

b) $\int \cos 2x\, dx = \frac{1}{2}\int 2\cos 2x = \frac{1}{2}\sin 2x$

19. a) $\int (-\sin x)\, dx = \cos x$

b) $\int \sin x\, dx = -\cos x$

20. a) $\int \sin (x - \pi)\, dx = -\cos (x - \pi)$

b) $\int \sin 2x\, dx = \frac{1}{2}\int 2\sin 2x = \frac{-1}{2}\cos 2x$

21. a) $\int (1 + tg^2\, 2x)\, dx = \frac{1}{2}\int 2(1 + tg^2\, 2x) = \frac{1}{2}\, tg\, 2x$

b) $\int tg^2\, 2x\, dx = \int (1 + tg^2\, 2x - 1) = \int (1 + tg^2\, 2x) - \int 1 = \frac{1}{2}\, tg\, 2x - x$

ALGUNES EXPONENCIALS

22. a) $\int e^x\, dx = e^x$

b) $\int e^{x+1}\, dx = e^{x+1}$

23. a) $\int e^{2x}\, dx = \frac{1}{2}\int 2e^{2x} = \frac{1}{2}\, e^{2x}$

b) $\int e^{2x+1}\, dx = \frac{1}{2}\int 2e^{2x+1} = \frac{1}{2} e^{2x+1}$

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1. Calcula les integrals següents:

a) $\int 7x^4\,dx$ **b)** $\int \frac{1}{x^2}\,dx$ **c)** $\int \sqrt{x}\,dx$

d) $\int \sqrt[3]{5x^2}\,dx$ **e)** $\int \frac{\sqrt[3]{x}+\sqrt{5x^3}}{3x}\,dx$ **f)** $\int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}}\,dx$

a) $\int 7x^4\,dx = 7\frac{x^5}{5} + k = \frac{7x^5}{5} + k$

b) $\int \frac{1}{x^2}\,dx = \int x^{-2} = \frac{x^{-1}}{-1} + k = \frac{-1}{x} + k$

c) $\int \sqrt{x}\,dx = \int x^{1/2} = \frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{x^3}}{3} + k$

d) $\int \sqrt[3]{5x^2}\,dx = \int \sqrt[3]{5}\,x^{2/3} = \sqrt[3]{5}\,\frac{x^{5/3}}{5/3} + k = \frac{3\sqrt[3]{5x^5}}{5} + k$

e) $\int \frac{\sqrt[3]{x}+\sqrt{5x^3}}{3x}\,dx = \int \frac{x^{1/3}}{3x} + \int \frac{\sqrt{5}\,x^{3/2}}{3x} = \frac{1}{3}\int x^{-2/3} + \frac{\sqrt{5}}{3}\int x^{1/2} =$

$= \frac{1}{3}\,\frac{x^{1/3}}{1/3} + \frac{\sqrt{5}}{3}\,\frac{x^{3/2}}{3/2} + k = \sqrt[3]{x} + \frac{2\sqrt{5x^3}}{9} + k$

f) $\int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}}\,dx = \int \frac{\sqrt{5}\cdot x^{3/2}}{\sqrt[3]{3}\cdot x^{1/3}} = \frac{\sqrt{5}}{\sqrt[3]{3}}\int x^{7/6} = \frac{\sqrt{5}}{\sqrt[3]{3}}\,\frac{x^{13/6}}{13/6} + k = \frac{6\sqrt{5}\,\sqrt[6]{x^{13}}}{13\sqrt[3]{3}} + k$

2. Calcula:

a) $\int \frac{x^4-5x^2+3x-4}{x}\,dx$ **b)** $\int \frac{x^4-5x^2+3x-4}{x+1}\,dx$

c) $\int \frac{x^4-5x^2+3x-4}{x^2+1}\,dx$ **d)** $\int \frac{x^3}{x-2}\,dx$

a) $\int \frac{x^4-5x^2+3x-4}{x}\,dx = \int\left(x^3-5x+3-\frac{4}{x}\right) = \frac{x^4}{4} - \frac{5x^2}{2} + 3x - 4\,ln\,|x| + k$

b) $\int \frac{x^4-5x^2+3x-4}{x+1}\,dx = \int\left(x^3-x^2-4x+7-\frac{11}{x+1}\right) =$

$= \frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 7x - 11\,ln\,|x+1| + k$

c) $\int \frac{x^4-5x^2+3x-4}{x^2+1}\,dx = \int\left(x^2-6+\frac{3x+2}{x^2+1}\right) = \int\left(x^2-6+\frac{3x}{x^2+1}+\frac{2}{x^2+1}\right) =$

$= \int x^2 - \int 6 + \frac{3}{2}\int\frac{2x}{x^2+1} + 2\int\frac{1}{x^2+1} =$

$= \frac{x^3}{3} - 6x + \frac{3}{2}\,ln\,(x^2+1) + 2\,arc\,tg\,x + k$

d) $\int \frac{x^3}{x-2}\, dx = \int \left(x^2 + 2x + 4 + \frac{8}{x-2}\right) = \frac{x^3}{3} + x^2 + 4x + 8\, ln\, |x-2| + k$

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3. **a)** $\int (3x - 5tg\, x)\, dx$ **b)** $\int (5cos\, x + 3^x)\, dx$

c) $\int (3tg\, x - 5cos\, x)\, dx$ **d)** $\int (10^x - 5^x)\, dx$

a) $\frac{3x^2}{2} - 5ln(cos\, x) + K$ b) $5sin\, x + \frac{3^x}{ln\, 3} + K$

c) $-3ln(cos\, x) - 5sin\, x + K$ d) $\frac{10^x}{ln\, 10} - \frac{5^x}{ln\, 5} + K$

4. **a)** $\int \frac{3}{x^2+1}\, dx$ **b)** $\int \frac{2x}{x^2+1}\, dx$ **c)** $\int \frac{x^2-1}{x^2+1}\, dx$ **d)** $\frac{(x-1)^2}{x^2+1}\, dx$

a) $3arctg\, x + K$ b) $ln(x^2+1) + K$ c) $x - 2arctg\, x$ d) $x - ln(x^2+1) + K$

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5. **Calcula:**

a) $\int cos^4\, x\, sin\, x\, dx$ **b)** $\int 2^{sin\, x} cos\, x\, dx$

a) $\int cos^4\, x\, sin\, x\, dx = -\int cos^4\, x(-sin\, x)\, dx = -\frac{cos^5\, x}{5} + k$

b) $\int 2^{sin\, x} cos\, x\, dx = \frac{1}{ln\, 2}\int 2^{sin\, x} cos\, x \cdot ln\, 2\, dx = \frac{2^{sin\, x}}{ln\, 2} + k$

6. **Calcula:**

a) $\int cotg\, x\, dx$ **b)** $\int \frac{5x}{x^4+1} dx$

a) $\int cotg\, x\, dx = \int \frac{cos\, x}{sin\, x}\, dx = ln\, |sin\, x| + k$

b) $\int \frac{5x}{x^4+1}\, dx = \frac{5}{2}\int \frac{2x}{1+(x^2)^2}\, dx = \frac{5}{2}\, arc\, tg\, (x^2) + k$

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7. **Calcula:** $\int x\, sin\, x\, dx$

$-x \cdot cos\, x + sin\, x + K$

8. Calcula: $\int x \; arctg \; x \; dx$

$x \cdot arctg\, x - ln(1 + x^2) + K$

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9. Calcula $\int x^4 e^x$

$$\int x^4 \cdot e^x = x^4 \cdot e^x - \int 4x^3 \cdot e^x\, dx = x^4 \cdot e^x - 4\int x^3 \cdot e^x \; dx =$$

$$= x^4 \cdot e^x - 4\,[x^3 \cdot e^x - 3x^2 \cdot e^x + 6x \cdot e^x - 6e^x] + k = e^x\,(x^4 - 4x^3 + 3x^2 - 6x + 6) + k$$

10. Calcula $\int sin^4 x\, dx$**. Calcula prèviament** $\int sin^2 x\, dx$**.**

Fem primer $\int sin^2\, x\, dx$

$$\int sin^2\, x\, dx = \int sin\, x \cdot sin\, x\, dx = -sin\, x\, cos\, x - \int -cos^2\, x\, dx = -sin\, x\, cos + \int cos^2\, x\, dx =$$

$u = sin\, x$	$v = cos\, x$
$u' = cos\, x$	$v' = sin\, x$

$$= -sin\, x\, cos\, x + \int (1 - sin^2\, x)\, dx = -sin\, x\, cos\, x + \int dx - \int sin^2\, x\, dx =$$

$$= x - sin\, x\, cos\, x - \int sin^2\, x\, dx$$

Si igualem els dos termes:

$$\int sin^2\, x\, dx = x - sin\, x\, cos\, x - \int sin^2\, x\, dx$$

$$2\int sin^2\, x\, dx = x - sin\, x\, cos\, x + k$$

$$\int sin^2\, x\, dx = \frac{x - sin\, x\, cos\, x}{2} + k = \frac{x}{2} - \frac{sin\, 2x}{4} + k$$

$sin\, 2x = 2sin\, x\, cos\, x$

Ara fem $\int sin^4\, x\, dx$

$$\int sin^4\, x\, dx = \int sin^2\, x\,(1 - cos^2\, x)\, dx = \int sin^2\, x\, dx - \int sin^2\, x\, cos^2\, x\, dx = \int sin^2\, x\, dx -$$

$sin\, 2x$	$= 2sin\, x\, cos\, x$
$sin^2\, 2x$	$= 4sin^2\, x\, cos^2\, x$

$$-\int \frac{1}{4} \cdot sin^2\, 2x\, dx = \int sin^2\, x\, dx - \frac{1}{4}\int sin^2\, 2x\, dx = \frac{x}{2} - \frac{sin\, 2x}{4} - \frac{1}{4}\left[\frac{x}{2} - \frac{sin\, 4x}{8}\right] k =$$

$$= \frac{3x}{8} - \frac{sin\, 2x}{4} + \frac{sin\, 4x}{32} + k$$

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11. **Calcula:** $\displaystyle\int \frac{3x^2 - 5x + 1}{x - 4}\,dx$

$$\int \frac{3x^2 - 5x + 1}{x - 4}\,dx = \int \left(3x + 7 + \frac{29}{x - 4}\right) dx = \frac{3x^2}{2} + 7x + 29\ ln\ |x - 4| + k$$

12. **Calcula:** $\displaystyle\int \frac{3x^2 - 5x + 1}{2x + 1}\,dx$

$$\int \frac{3x^2 - 5x + 1}{2x + 1}\,dx = \int \left(\frac{3}{2}x - \frac{13}{4} + \frac{17/4}{2x + 1}\right) dx =$$

$$= \frac{3}{2} \cdot \frac{x^2}{2} - \frac{13}{4}x - \frac{17}{8}\ ln\ |2x + 1| + k = \frac{3x^2}{4} - \frac{13}{4}x - \frac{17}{8}\ ln\ |2x + 1| + k$$

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13. **Calcula:**

a) $\displaystyle\int \frac{5x - 3}{x^3 - x}\,dx$ **b)** $\displaystyle\int \frac{x^2 - 2x + 6}{(x - 1)^3}\,dx$

a) Descomponem la fracció:

$$\frac{5x - 3}{x^3 - x} = \frac{5x - 3}{x(x - 1)(x + 1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}$$

$$\frac{5x - 3}{x^3 - x} = \frac{A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1)}{x(x - 1)(x + 1)}$$

$$5x - 3 = A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1)$$

Busquem A, B i C donant a x els valors 0, 1 i –1:

$$\left.\begin{array}{lllll} x = 0 & \Rightarrow & -3 = -A & \Rightarrow & A = 3 \\ x = 1 & \Rightarrow & 2 = 2B & \Rightarrow & B = 1 \\ x = -1 & \Rightarrow & -8 = 2C & \Rightarrow & C = -4 \end{array}\right\}$$

Així doncs, tenim que:

$$\int \frac{5x - 3}{x^3 - x}\,dx = \int \left(\frac{3}{x} + \frac{1}{x - 1} - \frac{4}{x + 1}\right) dx = 3\ ln|x| + ln|x - 1| - 4\ ln|x + 1| + k$$

b) Descomponem la fracció:

$$\frac{x^2 - 2x + 6}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3} = \frac{A(x - 1)^2 + B(x - 1) + C}{(x - 1)^3}$$

$$x^2 - 2x + 6 = A(x - 1)^2 + B(x - 1) + C$$

Si donem a x els valors 1, 0 i 2, queda:

$$\left.\begin{array}{lll} x = 1 & \Rightarrow & 5 = C \\ x = 0 & \Rightarrow & 6 = A - B + C \\ x = 2 & \Rightarrow & 6 = A + B + C \end{array}\right\} \begin{array}{l} A = 1 \\ B = 0 \\ C = 5 \end{array}$$

Per tant:

$$\int \frac{x^2 - 2x + 6}{(x-1)^3}\, dx = \int \left(\frac{1}{x-1} + \frac{5}{(x-1)^3} \right) dx = \ln|x-1| - \frac{5}{2(x-1)^2} + k$$

14. Calcula:

a) $\displaystyle\int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2}\, dx$ **b)** $\displaystyle\int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x}\, dx$

a) $x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x-2)(x+2)$

Descomponem la fracció:

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{x+2}$$

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x-2)(x+2)} =$$

$$= \frac{Ax(x-2)(x+2) + B(x-2)(x+2) + Cx^2(x+2) + Dx^2(x-2)}{x^2(x-2)(x+2)}$$

$$x^3 + 22x^2 - 12x + 8 = Ax(x-2)(x+2) + B(x-2)(x+2) + Cx^2(x+2) + Dx^2(x-2)$$

Busquem A, B, C i D donant a x els valors 0, 2, –2 i 1:

$$\left.\begin{array}{lllllllll} x = 0 & \Rightarrow & 8 = -4B & \Rightarrow & B = -2 \\ x = 2 & \Rightarrow & 80 = 16C & \Rightarrow & C = 5 \\ x = -2 & \Rightarrow & 112 = -16D & \Rightarrow & D = -7 \\ x = 1 & \Rightarrow & 19 = -3A - 3B + 3C - D & \Rightarrow & -3A = -9 & \Rightarrow & A = 3 \end{array}\right\}$$

Per tant:

$$\int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2}\, dx = \int \left(\frac{3}{x} - \frac{2}{x^2} + \frac{5}{x-2} - \frac{7}{x+2} \right) dx =$$

$$= 3 \ln|x| + \frac{2}{x} + 5 \ln|x-2| - 7 \ln|x+2| + k$$

b) La fracció es pot simplificar:

$$\frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} = \frac{x(x-2)^2}{x(x-2)^2(x+2)} = \frac{1}{x+2}$$

$$\int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x}\, dx = \int \frac{1}{x+2}\, dx = \ln|x+2| + k$$

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EXERCICIS I PROBLEMES PROPOSATS

PER PRACTICAR

15. **Calcula les següents integrals immediates:**

a) $\int (4x^2 - 5x + 7)dx$ **b)** $\int \frac{dx}{\sqrt[5]{x}}$ **c)** $\int \frac{1}{2x + 7}dx$ **d)** $\int (x - sin\ x)dx$

a) $\int (4x^2 - 5x + 7)\,dx = \frac{4x^3}{3} - \frac{5x^2}{2} + 7x + k$

b) $\int \frac{dx}{\sqrt[5]{x}} = \int x^{-1/5}\,dx = \frac{x^{4/5}}{4/5} + k = \frac{5\sqrt[5]{x^4}}{4} + k$

c) $\int \frac{1}{2x + 7}\,dx = \frac{1}{2}\,ln|2x + 7| + k$

d) $\int (x - sin\ x)\,dx = \frac{x^2}{2} + cos\ x + k$

16. **Resol aquestes integrals:**

a) $\int (x^2 + 4x)(x^2 - 1)dx$ **b)** $\int (x - 1)^3\,dx$

c) $\int \sqrt{3x}\,dx$ **d)** $\int (sin\ x + e^x)dx$

a) $\int (x^2 + 4x)(x^2 - 1)\,dx = \int (x^4 + 4x^3 - x^2 - 4x)\,dx = \frac{x^5}{5} + x^4 - \frac{x^3}{3} - 2x^2 + k$

b) $\int (x - 1)^3\,dx = \frac{(x - 1)^4}{4} + k$

c) $\int \sqrt{3x}\,dx = \int \sqrt{3}\,x^{1/2}\,dx = \sqrt{3}\,\frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{3x^3}}{3} + k$

d) $\int (sin\ x + e^x)\,dx = -cos\ x + e^x + k$

17. **Calcula les integrals següents:**

a) $\int \sqrt[3]{\frac{x}{2}}\,dx$ **b)** $\int sin\,(x - 4)dx$ **c)** $\int \frac{7}{cos^2 x}dx$ **d)** $\int (e^x + 3e^{-x})dx$

a) $\int \sqrt[3]{\frac{x}{2}}\,dx = \frac{1}{\sqrt[3]{2}}\int x^{1/3}\,dx = \frac{1}{\sqrt[3]{2}}\,\frac{x^{4/3}}{4/3} + k = \frac{3}{4}\sqrt[3]{\frac{x^4}{2}} + k$

b) $\int sin(x - 4)\,dx = -cos\,(x - 4) + k$

c) $\int \frac{7}{\cos^2 x}\, dx = 7\, tg\, x + k$

d) $\int (e^x + 3e^{-x})\, dx = e^x - 3e^{-x} + k$

18. **Troba aquestes integrals:**

a) $\int \frac{2}{x} dx$ **b)** $\int \frac{dx}{x-1}$ **c)** $\int \frac{x+\sqrt{x}}{x^2} dx$ **d)** $\int \frac{3}{1+x^2} dx$

a) $\int \frac{2}{x}\, dx = 2\, ln|x| + k$

b) $\int \frac{dx}{x-1} = ln|x-1| + k$

c) $\int \frac{x+\sqrt{x}}{x^2}\, dx = \int \left(\frac{1}{x} + x^{-3/2}\right) dx = ln|x| - \frac{2}{\sqrt{x}} + k$

d) $\int \frac{3}{1+x^2}\, dx = 3\, arc\, tg\, x + k$

19. **Resol les integrals següents:**

a) $\int \frac{dx}{x-4}$ **b)** $\int \frac{dx}{(x-4)^2}$ **c)** $\int (x-4)^2 dx$ **d)** $\int \frac{dx}{(x-4)^3}$

a) $\int \frac{dx}{x-4} = ln|x-4| + k$

b) $\int \frac{dx}{(x-4)^2} = \frac{-1}{(x-4)} + k$

c) $\int (x-4)^2\, dx = \frac{(x-4)^3}{3} + k$

d) $\int \frac{dx}{(x-4)^3} = \int (x-4)^{-3}\, dx = \frac{(x-4)^{-2}}{-2} + k = \frac{-1}{2(x-4)^2} + k$

20. **Troba les següents integrals del tipus exponencial:**

a) $\int e^{x-4} dx$ **b)** $\int e^{-2x+9} dx$ **c)** $\int e^{5x} dx$ **d)** $\int (3^x - x^3) dx$

a) $\int e^{x-4}\, dx = e^{x-4} + k$

b) $\int e^{-2x+9}\, dx = \frac{-1}{2}\int -2e^{-2x+9}\, dx = \frac{-1}{2} e^{-2x+9} + k$

c) $\int e^{5x}\,dx = \frac{1}{5}\int 5e^{5x}\,dx = \frac{1}{5}\,e^{5x} + k$

d) $\int (3^x - x^3)\,dx = \frac{3^x}{\ln 3} - \frac{x^4}{4} + k$

21. **Resol les següents integrals del tipus arc tangent:**

a) $\int \frac{dx}{4 + x^2}$ **b)** $\int \frac{4\,dx}{3 + x^2}$ **c)** $\int \frac{5\,dx}{4x^2 + 1}$ **d)** $\int \frac{2\,dx}{1 + 9x^2}$

a) $\int \frac{dx}{4 + x^2} = \int \frac{1/4}{1 + (x/2)^2}\,dx = \frac{1}{2}\int \frac{1/2}{1 + (x/2)^2}\,dx = \frac{1}{2}\,arc\,tg\left(\frac{x}{2}\right) + k$

b) $\int \frac{4\,dx}{3 + x^2} = \int \frac{4/3}{1 + (x/\sqrt{3})^2}\,dx = \frac{4\sqrt{3}}{3}\int \frac{1/\sqrt{3}}{1 + (x/\sqrt{3})^2}\,dx = \frac{4\sqrt{3}}{3}\,arc\,tg\left(\frac{x}{\sqrt{3}}\right) + k$

c) $\int \frac{5\,dx}{4x^2 + 1} = \frac{5}{2}\int \frac{2\,dx}{(2x)^2 + 1} = \frac{5}{2}\,arc\,tg\,(2x) + k$

d) $\int \frac{2\,dx}{1 + 9x^2} = \frac{2}{3}\int \frac{3\,dx}{1 + (3x)^2} = \frac{2}{3}\,arc\,tg\,(3x) + k$

22. **Expressa les integrals següents de la forma:**

$$\frac{\textbf{dividend}}{\textbf{divisor}} = \textbf{quocient} + \frac{\textbf{residu}}{\textbf{divisor}}$$

I les resols:

a) $\int \frac{x^2 - 5x + 4}{x + 1}\,dx$ **b)** $\int \frac{x^3 - 3x^2 + x - 1}{x - 2}\,dx$

a) $\int \frac{x^2 - 5x + 4}{x + 1}\,dx = \int \left(x - 6 + \frac{10}{x + 1}\right)dx = \frac{x^2}{2} - 6x + 10\,\ln|x + 1| + k$

b) $\int \frac{x^3 - 3x^2 + x - 1}{x - 2}\,dx = \int \left(x^2 - x - 1 - \frac{3}{x - 2}\right)dx =$

$= \frac{x^3}{3} - \frac{x^2}{2} - x - 3\,\ln|x - 2| + k$

23. **Troba aquestes integrals sabent que són del tipus arc sinus:**

a) $\int \frac{dx}{\sqrt{1 - 4x^2}}$ **b)** $\int \frac{dx}{\sqrt{4 - x^2}}$ **c)** $\int \frac{e^x}{\sqrt{1 - e^{2x}}}\,dx$ **d)** $\int \frac{dx}{x\sqrt{1 - (\ln x)^2}}$

a) $\int \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{1 - (2x)^2}} = \frac{1}{2}\,arc\,sin\,(2x) + k$

b) $\int \frac{dx}{\sqrt{4 - x^2}} = \int \frac{1/2\,dx}{\sqrt{1 - (x/2)^2}} = arc\,sin\left(\frac{x}{2}\right) + k$

c) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx = arc\,sin\,(e^x) + k$

d) $\int \frac{dx}{x\sqrt{1-(ln\ x)^2}} = \int \frac{1/x\ dx}{\sqrt{1-(ln\ x)^2}} = arc\,sin\,(ln|x|) + k$

24. **Resol les integrals següents, sabent que són de la forma $\int f^n(x)\cdot f'(x)$:**

a) $\int cos\ x\ sin^3 x\ dx$ **b)** $\int 2x\ e^{x^2}\,dx$ **c)** $\int \frac{x\ dx}{(x^2+3)^5}$ **d)** $\int \frac{1}{x} ln^3 x\ dx$

a) $\int cos\ x\ sin^3\ x\ dx = \frac{sin^4\ x}{4} + k$

b) $\int 2x\ e^{x^2}\ dx = e^{x^2} + k$

c) $\int \frac{x\ dx}{(x^2+3)^5} = \frac{1}{2}\int 2x(x^2+3)^{-5}\ dx = \frac{1}{2}\ \frac{(x^2+3)^{-4}}{-4} + k = \frac{-1}{8(x^2+3)^4} + k$

d) $\int \frac{1}{x}\ ln^3\ x\ dx = \frac{ln^4|x|}{4} + k$

25. **Resol les integrals següents:**

a) $\int x^4 e^{x^5}\,dx$ **b)** $\int x\ sin\ x^2\ dx$ **c)** $\int \frac{dx}{\sqrt{9-x^2}}$ **d)** $\int \frac{x\ dx}{\sqrt{x^2+5}}$

a) $\int x^4\ e^{x^5}\ dx = \frac{1}{5}\int 5x^4\ e^{x^5}\ dx = \frac{1}{5}\ e^{x^5} + k$

b) $\int x\ sin\ x^2\ dx = \frac{1}{2}\int 2x\ sin\ x^2\ dx = \frac{-1}{2}\ cos\ x^2 + k$

c) $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{1/3\ dx}{\sqrt{1-(x/3)^2}} = arc\ sin\left(\frac{x}{3}\right) + k$

d) $\int \frac{x\ dx}{\sqrt{x^2+5}} = \sqrt{x^2+5} + k$

26. **Resol les integrals següents:**

a) $\int sin\ x\ cos\ x\ dx$ **b)** $\int \frac{sin\ x\ dx}{cos^5\ x}$ **c)** $\int \sqrt{(x+3)^5}\ dx$ **d)** $\int \frac{-3x}{2-6x^2}dx$

a) $\int sin\ x\ cos\ x\ dx = \frac{sin^2\ x}{2} + k$

b) $\int \frac{sin\ x\ dx}{cos^5\ x} = -\int (-sin\ x)\cdot cos^{-5}\ x\ dx = \frac{-cos^{-4}\ x}{-4} + k = \frac{1}{4\ cos^4\ x} + k$

c) $\int \sqrt{(x+3)^5}\, dx = \int (x+3)^{5/2}\, dx = \frac{(x+3)^{7/2}}{7/2} + k = \frac{2\sqrt{(x+3)^7}}{7} + k$

d) $\int \frac{-3x}{2-6x^2}\, dx = \frac{1}{4}\int \frac{-12x}{2-6x^2}\, dx = \frac{1}{4}\, ln|2-6x^2| + k$

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27. Aplica la integració per parts per resoldre les integrals següents:

a) $\int x\, ln\, x\, dx$ **b)** $\int 3x \cos x\, dx$ **c)** $\int arc\, tg\, x\, dx$ **d)** $\int x\, e^{-x}\, dx$

a) $\int x\, ln\, x\, dx$

$$\begin{cases} u = ln\, x \;\to\; du = \frac{1}{x}\, dx \\ dv = x\, dx \;\to\; v = \frac{x^2}{2} \end{cases}$$

$$\int x\, ln\, x\, dx = \frac{x^2}{2}\, ln\, x - \int \frac{x}{2}\, dx = \frac{x^2}{2}\, ln|x| - \frac{x^2}{4} + k$$

b) $\int 3x \cos x\, dx$

$$\int 3x \cdot \cos x\, dx = 3\int x \cdot \cos x\, dx = x \cdot \sin x - \int \sin x\, dx = x \sin x + \cos x + k$$

$$\begin{array}{|ll|} \hline u = x & v = \sin x \\ u' = 1 & v' = \cos x \\ \hline \end{array}$$

c) $\int arc\, tg\, x\, dx$

$$\begin{cases} u = arc\, tg\, x \;\to\; du = \frac{1}{1+x^2}\, dx \\ dv = dx \;\to\; v = x \end{cases}$$

$$\int arc\, tg\, x = x\, arc\, tg\, x - \int \frac{1}{1+x^2}\, dx = x\, arc\, tg\, x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx =$$

$$= x\, arc\, tg\, x - \frac{1}{2}\, ln\,(1+x^2) + k$$

d) $\int x\, e^{-x}\, dx$

$$\int x \cdot e^{-x}\, dx = x \cdot (-e^{-x}) - \int -e^{-x}\, dx = -x \cdot e^{-x} - e^{-x} + k = e^{-x}(-x-1) + k$$

$$\begin{array}{|ll|} \hline u = x & v = e^{-x} \\ u' = 1 & v' = e^{-x} \\ \hline \end{array}$$

28. Resol les integrals següents aplicant dues vegades la integració per parts:

a) $\int e^x \cos x\, dx$ **b)** $\int x^2 \sin x\, dx$ **c)** $\int x^2\, e^{2x}$ **d)** $\int \cos\,(ln\, x)\, dx$

a) $\int e^x \cdot \cos x\, dx = e^x \cdot \sin x - \int e^x \cdot \sin x\, dx = e^x \cdot \sin x - \left[e^x(-\cos x) - \int e^x \cdot (-\cos x) dx\right] =$

$\boxed{\begin{array}{ll} u = e^x & v = \sin x \\ u' = e^x & v' = \cos x \end{array}}$ $\boxed{\begin{array}{ll} u = e^x & v = -\cos x \\ u' = e^x & v' = \sin x \end{array}}$

$= e^x \sin x + e^x \cdot \cos x - \int e^x \cdot \cos x\, dx + k$

Agafant la primera i l'última de les igualtats, construïm:

$\int e^x \cos x\, dx = e^x(\sin x + \cos x) - \int e^x \cos x\, dx + k$

$2\int e^x \cos x\, dx = e^x(\sin x + \cos x) + k$

$\int e^x \cos x\, dx = \frac{e^x}{2}(\sin x + \cos x) + k$

b) $\int x^2 \sin x\, dx = x^2(-\cos x) - \int 2x \cdot (-\cos x)\, dx = -x^2 \cos x + 2\int x \cdot \cos x\, dx =$

$\boxed{\begin{array}{ll} u = x^2 & v = -\cos x \\ u' = 2x & v' = \sin x \end{array}}$ $\boxed{\begin{array}{ll} u = x & v = \sin x \\ u' = 1 & v' = \cos x \end{array}}$

$= -x^2 \cos x + 2\left[x \sin x - \int \sin x\, dx\right] = -x^2 \cos x + 2x \sin x + 2x \cos x + k$

c) $\int x^2 \cdot e^{2x}\, dx = x^2 \cdot \frac{1}{2} e^{2x} - \int 2x \cdot \frac{1}{2} e^{2x}\, dx = \frac{x^2 e^x}{2} - \int x \cdot e^{2x}\, dx = \frac{x^2 e^{2x}}{2} -$

$\boxed{\begin{array}{ll} u = x^2 & v = \frac{1}{2} e^{2x} \\ u' = 2x & v' = e^{2x} \end{array}}$ $\boxed{\begin{array}{ll} u = x & v = \frac{1}{2} e^{2x} \\ u' = 1 & v' = e^{2x} \end{array}}$

$-\left[x \cdot \frac{e^{2x}}{2} - \int \frac{1}{2} e^{2x}\, dx\right] = \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{1}{2}\int e^{2x}\, dx = \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{2} + k =$

$= \frac{e^{2x}}{2}\left(x^2 - x + \frac{1}{2}\right) + k$

d) $\int \cos(\ln x)\, dx = \int 1 \cdot \cos(\ln x)\, dx = x \cdot \cos(\ln x) - \int [\sin(\ln x) \cdot \frac{1}{x} \cdot x\, dx =$

$\boxed{\begin{array}{ll} u = \cos(\ln x) & v = x \\ u' = -\sin(\ln x) \cdot \frac{1}{x} & v' = x \end{array}}$

$= x \cos(\ln x) + \int \sin(\ln x)\, dx = x \cos(\ln x) + \int 1 \cdot \sin(\ln x)\, dx =$

$= x \cdot \cos(\ln x) + x \cdot \sin(\ln x) - \int \cos(\ln x)\, dx$

$\boxed{\begin{array}{ll} u = \sin(\ln x) & v = x \\ u' = \cos(\ln x) \cdot \frac{1}{x} & v' = x \end{array}}$

Construint una igualtat amb el que està marcat:

$\int \cos(\ln x)\, dx = x \cos(\ln x) + x \sin(\ln x) - \int \cos(\ln x)\, dx$

$2\int \cos(\ln x)\, dx = x \cos(\ln x) + x \sin(\ln x)$

$\int \cos(\ln x)\, dx = \frac{x \cos(\ln x)}{2} + \frac{x \sin(\ln x)}{2} + k$

29. **Resol les integrals que es proposen a continuació:**

a) $\int x \cdot 2^{-x}\,dx$ **b)** $\int arc\cos x\,dx$ **c)** $\int x\cos 3x\,dx$ **d)** $\int x^5 e^{-x^3}\,dx$

a) $\int x \cdot 2^{-x}\,dx$

$$\begin{cases} u = x \;\rightarrow\; du = dx \\ dv = 2^{-x}\,dx \;\rightarrow\; v = \dfrac{-2^{-x}}{\ln 2} \end{cases}$$

$$\int x\,2^{-x}\,dx = \frac{-x \cdot 2^{-x}}{\ln 2} + \int \frac{2^{-x}}{\ln 2}\,dx = \frac{-x \cdot 2^{-x}}{\ln 2} + \frac{1}{\ln 2}\int 2^{-x}\,dx =$$
$$= \frac{-x \cdot 2^{-x}}{\ln 2} - \frac{2^{-x}}{(\ln 2)^2} + k$$

b) $\int arc\cos x\,dx$

$$\begin{cases} u = arc\cos x \;\rightarrow\; du = \dfrac{-1}{\sqrt{1-x^2}}\,dx \\ dv = dx \;\rightarrow\; v = x \end{cases}$$

$$\int arc\cos x\,dx = x\,arc\cos x - \int \frac{-x}{\sqrt{1-x^2}}\,dx = x\,arc\cos x - \sqrt{1-x^2} + k$$

c) $\int x\cos 3x\,dx$

$$\begin{cases} u = x \;\rightarrow\; du = dx \\ dv = \cos 3x\,dx \;\rightarrow\; v = \dfrac{1}{3}\sin 3x \end{cases}$$

$$\int x\cos 3x\,dx = \frac{x}{3}\sin 3x - \frac{1}{3}\int \sin 3x\,dx = \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + k$$

d) $\int x^5 e^{-x^3}\,dx = \int \underbrace{x^3}_{u} \cdot \underbrace{x^2 e^{-x^3}\,dx}_{dv}$

$$\begin{cases} u = x^3 \;\rightarrow\; du = 3x^2\,dx \\ dv = x^2 e^{-x^3}\,dx \;\rightarrow\; v = \dfrac{-1}{3}e^{-x^3} \end{cases}$$

$$\int x^5 e^{-x^3}\,dx = \frac{-x^3}{3}e^{-x^3} + \int x^2 e^{-x^3}\,dx = \frac{-x^3}{3}e^{-x^3} - \frac{1}{3}e^{-x^3} + k =$$
$$= \frac{(-x^3-1)}{3}e^{-x^3} + k$$

30. **Aplica la descomposició en fraccions simples per resoldre les integrals següents:**

a) $\int \dfrac{1}{x^2+x-6}\,dx$ **b)** $\int \dfrac{3x^3}{x^2-4}\,dx$

c) $\int \dfrac{x^2+1}{x^2+x}\,dx$ **d)** $\int \dfrac{4}{x^2+x-2}\,dx$

e) $\displaystyle\int \frac{x^2}{x^2 + 4x + 3}\, dx$ **f)** $\displaystyle\int \frac{-16}{x^2 - 2x - 15}\, dx$

a) $\dfrac{1}{x^2 + x - 6} = \dfrac{1}{(x + 3)\,(x - 2)} = \dfrac{A}{x + 3} + \dfrac{B}{x - 2} = \dfrac{A(x - 2) + B(x + 3)}{(x + 3)\,(x - 2)} =$

$\dfrac{(A + B)x + (3B + 2A)}{(x + 3)\,(x - 2)}$

$$\begin{cases} A + B = 0 & A = -\dfrac{1}{5} \\ 3B + 2A = 1 & B = \dfrac{1}{5} \end{cases}$$

$$\int \frac{1}{x^2 + x - 6}\, dx = \int \frac{-1/5}{x + 3}\, dx + \int \frac{1/5}{x - 2}\, dx = -\frac{1}{5}\, ln\, |x + 3| + \frac{1}{5}\, ln\, |x - 2| + k$$

b) $\dfrac{3x^3}{x^2 - 4} = 3x + \dfrac{12x}{x^2 - 4} = 3x + \dfrac{12x}{(x - 2)\,(x + 2)} = 3x + \dfrac{A}{x - 2} + \dfrac{B}{x + 2}$

$$\begin{cases} A + B = 12 & A = 6 \\ 2A - 2B = 0 & B = 6 \end{cases}$$

$$\int \frac{3x^3}{x^2 - 4}\, dx = \int \left(3x + \frac{6}{x - 2} + \frac{6}{x + 2}\right) dx = \frac{3}{2}\, x^2 + 6ln\, |x - 2| + 6ln\, |x + 2| + k$$

c) $\dfrac{x^2 + 1}{x^2 + x} = 1 + \dfrac{1 - x}{x^2 + x} = 1 + \dfrac{1 - x}{x\,(x + 1)} = 1 + \dfrac{A}{x} + \dfrac{B}{x + 1}$

$$\begin{cases} A + B = -1 & A = 6 \\ A = 1 & B = -2 \end{cases}$$

$$\int \frac{x^2 + 1}{x^2 + x}\, dx = \int \left(1 + \frac{1}{x} + \frac{-2}{x + 1}\right) dx = x + ln\, |x| - 2ln\, |x + 1| + k$$

d) $\dfrac{4}{x^2 + x - 2} = \dfrac{4}{(x + 2)\,(x - 1)} = \dfrac{A}{x + 2} + \dfrac{B}{x - 1}$

$$\begin{cases} A + B = 0 & A = -\dfrac{4}{3} \\ 2B - A = 4 & B = \dfrac{4}{3} \end{cases}$$

$$\int \frac{4}{x^2 + x - 2}\, dx = \int \frac{-4/3}{x + 2}\, dx + \int \frac{4/3}{x - 1}\, dx = \frac{-4}{3}\, ln\, |x + 2| + \frac{4}{3}\, ln\, |x - 1| + k$$

e) $\dfrac{x^2}{x^2 + 4x + 3} = 1 + \dfrac{-4x - 3}{x^2 + 4x + 3} = 1 + \dfrac{-4x - 3}{(x + 3)\,(x + 1)} = 1 + \dfrac{A}{x + 3} + \dfrac{B}{x + 1}$

$$\begin{cases} A = -\dfrac{9}{2} \\ B = \dfrac{1}{2} \end{cases}$$

$$\int \frac{x^2}{x^2 + 4x + 3}\, dx = \int \left(1 + \frac{-9/2}{x + 3} + \frac{1/2}{x + 1}\right) dx = x - \frac{9}{2}\, ln\, |x + 3|\, \frac{1}{2}\, ln\, |x + 1| + k$$

f) $\dfrac{-16}{x^2-2x-15} = \dfrac{-16}{(x+3)(x-5)} = \dfrac{A}{x+3} + \dfrac{B}{x-5}$

$$\begin{cases} A + B = 0 \\ 3B - 5A = -16 \end{cases} \qquad \begin{matrix} A = 2 \\ B = -2 \end{matrix}$$

$$\int \frac{-16}{x^2-2x-15}\,dx = \int \left(\frac{2}{x+3} + \frac{-2}{x-5}\right) dx = 2ln\,|x+3| - 2ln\,|x-5| + k$$

31. **Resol les integrals següents:**

a) $\displaystyle\int \frac{2x-4}{(x-1)^2(x+3)}dx$

b) $\displaystyle\int \frac{2x+3}{(x-2)(x+5)}dx$

c) $\displaystyle\int \frac{1}{(x-1)(x+3)^2}dx$

d) $\displaystyle\int \frac{3x-2}{x^2-4}dx$

a) $\displaystyle\int \frac{2x-4}{(x-1)^2(x+3)}\,dx$

Descomponem en fraccions simples:

$$\frac{2x-4}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

$$\frac{2x-4}{(x-1)^2(x+3)} = \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2(x+3)}$$

$$2x-4 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

Busquem A, B i C:

$$\left.\begin{matrix} x = 1 & \rightarrow & -2 = 4B & \rightarrow & B = -1/2 \\ x = -3 & \rightarrow & -10 = 16C & \rightarrow & C = -5/8 \\ x = 0 & \rightarrow & -4 = -3A + 3B + C & \rightarrow & A = 5/8 \end{matrix}\right\}$$

Per tant:

$$\int \frac{2x-4}{(x-1)^2(x+3)}\,dx = \int \frac{5/8}{x-1}\,dx + \int \frac{-1/2}{(x-1)^2}\,dx + \int \frac{-5/8}{x+3}\,dx =$$

$$= \frac{5}{8}\,ln|x-1| + \frac{1}{2}\cdot\frac{1}{(x-1)} - \frac{5}{8}\,ln|x+3| + k = \frac{5}{8}\,ln\left|\frac{x-1}{x+3}\right| + \frac{1}{2x-2} + k$$

b) $\displaystyle\int \frac{2x+3}{(x-2)(x+5)}\,dx$

Descomponem en fraccions simples:

$$\frac{2x+3}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} = \frac{A(x+5) + B(x-2)}{(x-2)(x+5)}$$

$$2x+3 = A(x+5) + B(x-2)$$

Busquem A i B:

$$\left.\begin{array}{lll} x = 2 & \rightarrow \quad 7 = 7A & \rightarrow \quad A = 1 \\ x = -5 & \rightarrow \quad -7 = -7B & \rightarrow \quad B = 1 \end{array}\right\}$$

Per tant:

$$\int \frac{2x+3}{(x-2)\,(x+5)}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{1}{x+5}\,dx =$$

$$= ln|x-2| + ln|x+5| + k = ln|(x-2)(x+5)| + k$$

c) $\displaystyle\int \frac{1}{(x-1)\,(x+3)^2}\,dx$

Descomponem en fraccions simples:

$$\frac{1}{(x-1)\,(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

$$\frac{1}{(x-1)\,(x+3)^2} = \frac{A(x+3)^2 + B(x-1)(x+3) + C(x-1)}{(x-1)\,(x+3)^2}$$

$$1 = A(x+3)^2 + B(x-1)(x+3) + C(x-1)$$

Busquem A, B i C:

$$\left.\begin{array}{lll} x = 1 & \rightarrow \quad 1 = 16A & \rightarrow \quad A = 1/16 \\ x = -3 & \rightarrow \quad 1 = -4C & \rightarrow \quad C = -1/4 \\ x = 0 & \rightarrow \quad 1 = 9A - 3B - C & \rightarrow \quad B = -1/16 \end{array}\right\}$$

Per tant:

$$\int \frac{1}{(x-1)\,(x+3)^2}\,dx = \int \frac{1/16}{x-1}\,dx + \int \frac{-1/16}{x+3}\,dx + \int \frac{-1/4}{(x+3)^2}\,dx =$$

$$= \frac{1}{16}\,ln|x-1| - \frac{1}{16}\,ln|x+3| + \frac{1}{4}\cdot\frac{1}{(x+3)} + k =$$

$$= \frac{1}{16}\,ln\left|\frac{x-1}{x+3}\right| + \frac{1}{4(x+3)} + k$$

d) $\displaystyle\int \frac{3x-2}{x^2-4}\,dx = \int \frac{3x-2}{(x-2)\,(x+2)}\,dx$

Descomponem en fraccions simples:

$$\frac{3x-2}{(x-2)\,(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)\,(x+2)}$$

$$3x - 2 = A(x+2) + B(x-2)$$

Busquem A i B:

$$\left.\begin{array}{lll} x = 2 & \rightarrow \quad 4 = 4A & \rightarrow \quad A = 1 \\ x = -2 & \rightarrow \quad -8 = -4B & \rightarrow \quad B = 2 \end{array}\right\}$$

Per tant:

$$\int \frac{3x-2}{x^2-4}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{2}{x+2}\,dx =$$

$$= ln|x-2| + 2\,ln|x+2| + k = ln\left[|x-2|(x+2)^2\right] + k$$

32. **Resol:**

a) $\int \frac{2x^2+7x-1}{x^3+x^2-x-1}\,dx$ **b)** $\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx$

a) $\frac{2x^2+7x-1}{x^3+x^2-x-1} = \frac{2x^2+7x-1}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} =$

$= \frac{A(x+1)(x-1)+B(x-1)+C(x+1)^2}{(x+1)^2(x-1)} = \frac{Ax^2-A+Bx-B+Cx^2+2Cx+C}{(x+1)^2(x-1)}$

$$\begin{cases} A+C=2 & A=0 \\ B+2C=7 & B=3 \\ C-A-B=-1 & C=2 \end{cases}$$

$$\int \frac{2x^2+7x-1}{x^3+x^2-x-1}\,dx = \int\left(\frac{3}{(x+1)^2} + \frac{2}{x-1}\right)dx = \frac{-3}{x+1} + 2ln\,|x-1| + k$$

b) $\frac{2x^2+5x-1}{x^3+x^2-2x} = \frac{2x^2+5x-1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} =$

$= \frac{A(x-1)(x+2)+Bx(x+2)+Cx(x-1)}{x(x-1)(x+2)} = \frac{Ax^2+Ax-2A+Bx^2+2Bx+Cx^2-Cx}{x(x-1)(x+2)}$

$$\begin{cases} A+B+C=2 & A=1/2 \\ A+2B-C=5 & B=2 \\ -2A=-1 & C=-1/2 \end{cases}$$

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx = \int\left(\frac{1/2}{x} + \frac{2}{x-1} + \frac{-1/2}{x+2}\right)dx = \frac{1}{2}\,ln|x| + 2ln|x-1| - \frac{1}{2}\,ln|x+2| + k$$

33. **Resol les integrals següents:**

a) $\int \sqrt{x^2-2x}\,(x-1)\,dx$ **b)** $\int \frac{arc\,sin\,x}{\sqrt{1-x^2}}$

c) $\int \frac{(1+ln\,x)^2}{x}\,dx$ **d)** $\int \sqrt{(1+cos\,x)^3}\;sin\,x\,dx$

a) $\int \sqrt{x^2-2x}\,(x-1)\,dx = \frac{1}{2}\int \sqrt{x^2-2x}\,(2x-2)\,dx = \frac{1}{2}\int (x^2-2x)^{1/2}(2x-2)\,dx =$

$= \frac{1}{2}\,\frac{(x^2-2x)^{3/2}}{3/2} + k = \frac{\sqrt{(x^2-2x)^3}}{3} + k$

b) $\int \frac{arc\,sin\,x}{\sqrt{1-x^2}}\,dx = \int arc\,sin\,x \cdot \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\,arcsin^2\,x + K$

c) $\int \frac{(1+ln\,x)^2}{x}\,dx = \int (1+ln\,x)^2 \cdot \frac{1}{x}\,dx = \frac{(1+ln|x|)^3}{3} + k$

d) $\int \sqrt{(1 + cos\ x)^3}\ sin\ x\ dx = -\int (1 + cos\ x)^{3/2}\ (-sin\ x)\ dx = -\frac{(1 + cos\ x)^{5/2}}{5/2} + k =$

$= \frac{-2\sqrt{(1 + cos\ x)^5}}{5} + k$

34. **Per resoldre la integral $\int cos^3\ x\ dx$, farem $cos^3\ x = cos\ x\ cos^2\ x = cos\ x\ (1 - sin^2\ x) = cos\ x - cos\ x\ sin^2\ x$. La descomponem en dues integrals immediates. Resol·la.**

Calcula després: $\int sin^3\ x\ dx$

$$\int cos^3\ x\ dx = \int cos\ x\ dx - \int cos\ x \cdot sin^2\ x\ dx = sin\ x - \frac{sin^3\ x}{3} + k$$

$$\int sin^3\ x\ dx = \int sin\ x\,(sin^2\ x) dx = \int sin\ x\,\left[(1 - cos^2\ x)\right] dx = \int sin\ x\ dx - \int sin\ x\ cos^2\ x\ dx =$$

$$= -cos\ x + \frac{1}{3}\ cos^3\ x + k$$

35. **Calcula:**

a) $\int \frac{dx}{x^2 - x - 2}$

b) $\int \frac{x^4 + 2x - 6}{x^3 + x^2 - 2x} dx$

c) $\int \frac{5x^2}{x^3 - 3x^2 + 3x - 1} dx$

d) $\int \frac{2x - 3}{x^3 - 2x^2 - 9x + 18} dx$

a) $\int \frac{dx}{x^2 - x - 2} = \int \frac{dx}{(x + 1)\,(x - 2)}$

Descomponem en fraccions simples:

$$\frac{1}{(x + 1)\,(x - 2)} = \frac{A}{x + 1} + \frac{B}{x - 2} = \frac{A(x - 2) + B(x + 1)}{(x + 1)\,(x - 2)}$$

$$1 = A(x - 2) + B(x + 1)$$

Busquem A i B:

$$\left.\begin{array}{lll} x = -1 & \rightarrow\ 1 = -3A & \rightarrow\ A = -1/3 \\ x = 2 & \rightarrow\ 1 = 3B & \rightarrow\ B = 1/3 \end{array}\right\}$$

Per tant:

$$\int \frac{dx}{x^2 - x - 2}\ dx = \int \frac{-1/3}{x + 1}\ dx + \int \frac{1/3}{x - 2}\ dx =$$

$$= \frac{-1}{3}\ ln|x + 1| + \frac{1}{3}\ ln|x - 2| + k = \frac{1}{3}\ ln\left|\frac{x - 2}{x + 1}\right| + k$$

b) $\int \frac{x^4+2x-6}{x^3+x^2-2x}\,dx = \int \left(x-1+\frac{3x^2-6}{x(x-1)(x+2)} \right) dx$

Descomponem en fraccions simples:

$$\frac{3x^2-6}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$\frac{3x^2-6}{x(x-1)(x+2)} = \frac{A(x-1)(x+2)+Bx(x+2)+Cx(x-1)}{x(x-1)(x+2)}$$

$$3x^2-6 = A(x-1)(x+2)+Bx(x+2)+Cx(x-1)$$

Busquem A, B i C:

$$\left.\begin{array}{lllll} x=0 & \to & -6=-2A & \to & A=3 \\ x=1 & \to & -3=3B & \to & B=-1 \\ x=-2 & \to & 6=6C & \to & C=1 \end{array}\right\}$$

Per tant:

$$\int \frac{x^4+2x-6}{x^3+x^2-2x}\,dx = \int \left(x-1+\frac{3}{x}-\frac{1}{x-1}+\frac{1}{x+2} \right) dx =$$

$$= \frac{x^2}{2} - x + 3\,ln|x| - ln|x-1| + ln|x+2| + k =$$

$$= \frac{x^2}{2} - x + ln\left|\frac{x^3(x+2)}{x-1}\right| + k$$

c) $\int \frac{5x^2}{x^3-3x^2+3x-1}\,dx = \int \frac{5x^2}{(x-1)^3}\,dx$

Descomponem en fraccions simples:

$$\frac{5x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2+B(x-1)+C}{(x-1)^3}$$

$$5x^2 = A(x-1)^2+B(x-1)+C$$

Busquem A, B i C:

$$\left.\begin{array}{lll} x=1 & \to & 5=C \\ x=2 & \to & 20=A+B+C \\ x=0 & \to & 0=A-B+C \end{array}\right\} \begin{array}{l} A=5 \\ B=10 \\ C=5 \end{array}$$

Per tant:

$$\int \frac{5x^2}{x^3-3x^2+3x-1}\,dx = \int \left(\frac{5}{x-1} + \frac{10}{(x-1)^2} + \frac{5}{(x-1)^3} \right) dx =$$

$$= 5\,ln|x-1| - \frac{10}{x-1} - \frac{5}{2(x-1)^2} + k$$

d) $\displaystyle\int \frac{2x-3}{x^3-2x^2-9x+18}\,dx = \int \frac{2x-3}{(x-2)(x-3)(x+3)}\,dx$

Descomponem en fraccions simples:

$$\frac{2x-3}{(x-2)(x-3)(x+3)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x+3}$$

$$\frac{2x-3}{(x-2)(x-3)(x+3)} = \frac{A(x-3)(x+3) + B(x-2)(x+3) + C(x-2)(x-3)}{(x-2)(x-3)(x+3)}$$

$$2x-3 = A(x-3)(x+3) + B(x-2)(x+3) + C(x-2)(x-3)$$

Busquem A, B i C:

$$\left.\begin{array}{lllll} x=2 & \to & 1=-5A & \to & A=-1/5 \\ x=3 & \to & 3=6B & \to & B=1/2 \\ x=-3 & \to & -9=30C & \to & C=-3/10 \end{array}\right\}$$

Per tant:

$$\int \frac{2x-3}{x^3-2x^2-9x+18}\,dx = \int\left(\frac{-1/5}{x-2} + \frac{1/2}{x-3} + \frac{-3/10}{x+3}\right)dx =$$

$$= \frac{-1}{5}\ ln|x-2| + \frac{1}{2}\ ln|x-3| - \frac{3}{10}\ ln|x+3| + k$$

36. Resol les integrals:

a) $\displaystyle\int \frac{ln\, x}{x}\,dx$ **b)** $\displaystyle\int \frac{1-sin\, x}{x+cos\, x}\,dx$ **c)** $\displaystyle\int \frac{1}{x\, ln\, x}\,dx$

d) $\displaystyle\int \frac{1+e^x}{e^x+x}\,dx$ **e)** $\displaystyle\int \frac{sin\,(1/x)}{x^2}\,dx$ **f)** $\displaystyle\int \frac{2x-3}{x+2}\,dx$

g) $\displaystyle\int \frac{arc\, tg\, x}{1+x^2}\,dx$ **h)** $\displaystyle\int \frac{sin\, x}{cos^4\, x}\,dx$

a) $\displaystyle\int \frac{ln\, x}{x}\,dx = \int \frac{1}{x}\ ln\, x\,dx = \frac{ln^2|x|}{2} + k$

b) $\displaystyle\int \frac{1-sin\, x}{x+cos\, x}\,dx = ln|x+cos\, x| + k$

c) $\displaystyle\int \frac{1}{x\, ln\, x}\,dx = \int \frac{1/x}{ln\, x}\,dx = ln|\,ln|x|\,| + k$

d) $\displaystyle\int \frac{1+e^x}{e^x+x}\,dx = ln|e^x+x| + k$

e) $\int \frac{\sin(1/x)}{x^2}\,dx = -\int \frac{-1}{x^2}\sin\left(\frac{1}{x}\right)dx = \cos\left(\frac{1}{x}\right) + k$

f) $\int \frac{2x-3}{x+2}\,dx = \int\left(2 - \frac{7}{x+2}\right)dx = 2x - 7\ln|x+2| + k$

g) $\int \frac{\operatorname{arc\,tg} x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\operatorname{arc\,tg} x\,dx = \frac{\operatorname{arc\,tg}^2 x}{2} + k$

h) $\int \frac{\sin x}{\cos^4 x}\,dx = -\int(-\sin x)(\cos x)^{-4}\,dx = \frac{-(\cos x)^{-3}}{-3} + k = \frac{1}{3\cos^3 x} + k$

37. **Calcula $\int \cos^4 x\,dx$ utilitzant l'expressió: $\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$**

$$\cos^4 x = \left(\frac{1}{2} + \frac{\cos 2x}{2}\right)^2 = \frac{1}{4} + \frac{\cos^2 2x}{4} + \frac{\cos 2x}{2} =$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{\cos 4x}{2}\right) + \frac{\cos 2x}{2} =$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2} = \frac{3}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2}$$

Per tant:

$$\int \cos^4 x\,dx = \int\left(\frac{3}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2}\right)dx = \frac{3}{8}x + \frac{\sin 4x}{32} + \frac{\sin 2x}{2} + k$$

38. **Resol les integrals indefinides:**

a) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ **b)** $\int \ln(x-3)\,dx$ **c)** $\int \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$

d) $\int \ln(x^2+1)\,dx$ **e)** $\int (\ln x)^2\,dx$ **f)** $\int e^x \cos e^x\,dx$

g) $\int \frac{1}{1-x^2}\,dx$ **h)** $\int \frac{(1-x)^2}{1+x}\,dx$

a) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = -2\int \frac{1}{2\sqrt{x}}(-\sin\sqrt{x})\,dx = -2\cos(\sqrt{x}) + k$

b) $\int \ln(x-3)\,dx$

$$\begin{cases} u = ln\,(x-3) & \rightarrow \quad du = \dfrac{1}{x-3}\,dx \\ dv = dx & \rightarrow \quad v = x \end{cases}$$

$$\int ln\,(x-3)\,dx = x\,ln|x-3| - \int \frac{x}{x-3}\,dx = x\,ln|x-3| - \int 1 + \frac{3}{x-3}\,dx =$$

$$= x\,ln|x-3| - x - 3\,ln|x-3| + k = (x-3)\,ln|x-3| - x + k$$

c) $\displaystyle\int \frac{ln\sqrt{x}}{\sqrt{x}}\,dx$

$$\begin{cases} u = ln\sqrt{x} & \rightarrow \quad du = \dfrac{1}{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2x}\,dx \\ v = \dfrac{1}{\sqrt{x}}\,dx & \rightarrow \quad dv = 2\sqrt{x} \end{cases}$$

$$\int \frac{ln\sqrt{x}}{\sqrt{x}}\,dx = 2\sqrt{x}\,ln\sqrt{x} - \int \frac{2\sqrt{x}}{2x}\,dx = 2\sqrt{x}\,ln\sqrt{x} - \int \frac{1}{\sqrt{x}}\,dx =$$

$$= 2\sqrt{x}\,ln\sqrt{x} - 2\sqrt{x} + k = 2\sqrt{x}\,(ln\sqrt{x} - 1) + k$$

d) $\displaystyle\int ln\,(x^2+1)\,dx$

$$\begin{cases} u = ln\,(x^2+1) & \rightarrow \quad du = \dfrac{2x}{x^2+1}\,dx \\ dv = dx & \rightarrow \quad v = x \end{cases}$$

$$\int ln\,(x^2+1)\,dx = x\,ln\,(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx =$$

$$= x\,ln\,(x^2+1) - \int \left(2 - \frac{2}{x^2+1}\right)dx = x\,ln\,(x^2+1) - 2x + 2\;arc\;tg\;x + k$$

e) $\displaystyle\int (ln\,x)^2\,dx$

$$\begin{cases} u = (ln\,x)^2 & \rightarrow \quad du = 2\,(ln\,x) \cdot \dfrac{1}{x}\,dx \\ dv = dx & \rightarrow \quad v = x \end{cases}$$

$$\int (ln\,x)^2\,dx = x\,(ln\,x)^2 - 2\int ln\,x\,dx = x\,ln^2|x| - 2\,x\,ln|x| + 2x + k$$

f) $\displaystyle\int e^x\cos e^x\,dx = sin\,e^x + k$

g) $\displaystyle\int \frac{1}{1-x^2}\,dx = \int \frac{-1}{(x+1)(x-1)}\,dx$

Descomponem en fraccions simples:

$$\frac{-1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

Busquem A i B:

$$\left.\begin{array}{lll} x=-1 & \rightarrow\ -1=-2A & \rightarrow\ A=1/2 \\ x=1 & \rightarrow\ -1=2B & \rightarrow\ B=-1/2 \end{array}\right\}$$

Per tant:

$$\int \frac{1}{1-x^2}\,dx = \int\left(\frac{1/2}{x+1} + \frac{-1/2}{x-1}\right)dx =$$

$$= \frac{1}{2}\,ln|x+1| + \frac{1}{2}\,ln|x-1| + k = ln\sqrt{\frac{x+1}{x-1}} + k$$

h) $\displaystyle\int \frac{(1-x)^2}{1+x}\,dx = \int \frac{x^2-2x+1}{x+1}\,dx = \int\left(x-3+\frac{4}{x+1}\right)dx =$

$$= \frac{x^2}{2} - 3x + 4\,ln|x+1| + k$$

39. **Resol:**

a) $\displaystyle\int \frac{1}{1+e^x}\,dx$

☛ ***En el numerador, suma i resta e^x.***

b) $\displaystyle\int \frac{x+3}{\sqrt{9-x^2}}\,dx$

☛ ***Descompon-la en suma d'unes altres dues.***

a) $\displaystyle\int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx = \int\left(1-\frac{e^x}{1+e^x}\right) = x - ln(1+e^x) + k$

b) $\displaystyle\int \frac{x+3}{\sqrt{9-x^2}}\,dx = -\int \frac{-x}{\sqrt{9-x^2}}\,dx + \int \frac{3}{\sqrt{9-x^2}}\,dx =$

$$= -\sqrt{9-x^2} + 3\int \frac{1/3}{\sqrt{1-(x/3)^2}}\,dx = -\sqrt{9-x^2} + 3arc\,sin\left(\frac{x}{3}\right) + k$$

40. **Resol per substitució:**

a) $\displaystyle\int x\sqrt{x+1}\,dx$ **b)** $\displaystyle\int \frac{dx}{x-\sqrt[4]{x}}$ **c)** $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

d) $\displaystyle\int \frac{1}{x\sqrt{x+1}}\,dx$ **e)** $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$ **f)** $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$

☛ ***a) Fes $x + 1 = t^2$. b) Fes $x = t^4$. c) Fes $x = t^2$.***

a) $\int x\sqrt{x+1}\ dx$

Canvi: $x + 2 = t^2 \ \rightarrow\ dx = 2t\,dt$

$$\int x\sqrt{x+1}\ dx = \int (t^2-1)\,t \cdot 2t\,dt = \int (2t^4 - 2t^2)\ dt = \frac{2t^5}{5} - \frac{2t^3}{3} + k =$$

$$= \frac{2\sqrt{(x+1)^5}}{5} - \frac{2\sqrt{(x+1)^3}}{3} + k$$

b) $\int \frac{dx}{x - \sqrt[4]{x}}$

Canvi: $x = t^4 \ \rightarrow\ dx = 4t^3\,dt$

$$\int \frac{dx}{x-\sqrt[4]{x}} = \int \frac{4t^3\,dt}{t^4 - t} = \int \frac{4t^2\,dt}{t^3-1} = \frac{4}{3}\int \frac{3t^2\,dt}{t^3-1} = \frac{4}{3}\,ln\,|\,t^3 - 1\,| + k =$$

$$= \frac{4}{3}\,ln\,|\,\sqrt[4]{x^3} - 1\,| + k$$

c) $\int \frac{x}{\sqrt{x+1}}\ dx$

Canvi: $x + 1 = t^2 \ \rightarrow\ dx = 2t\,dt$

$$\int \frac{x}{\sqrt{x+1}}\ dx = \int \frac{(t^2-1)}{t} \cdot 2t\,dt = \int (2t^2 - 2)\ dt = \frac{2t^3}{3} - 2t + k =$$

$$= \frac{2\sqrt{(x+1)^3}}{3} - 2\sqrt{x+1} + k$$

d) $\int \frac{1}{x\ \sqrt{x+1}}\ dx$

Canvi: $x + 1 = t^2 \ \rightarrow\ dx = 2t\,dt$

$$\int \frac{1}{x\ \sqrt{x+1}}\ dx = \int \frac{2t\ dt}{(t^2-1)\,t} = \int \frac{2\ dt}{(t+1)(t-1)}$$

Descomponem en fraccions simples:

$$\frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} = \frac{A(t-1) + B(t+1)}{(t+1)(t-1)}$$

$2 = A(t-1) + B(t+1)$

Busquem A i B:

$$\left.\begin{array}{lllll} t = -1 & \rightarrow & 2 = -2A & \rightarrow & A = -1 \\ t = 1 & \rightarrow & 2 = 2B & \rightarrow & B = 1 \end{array}\right\}$$

Per tant:

$$\int \frac{2\ dt}{(t+1)(t-1)} = \int \left(\frac{-1}{t+1} + \frac{1}{t-1} \right) dt = -ln|t+1| + ln|t-1| + k =$$

$$= ln\left| \frac{t-1}{t+1} \right| + k$$

Així doncs:

$$\int \frac{1}{x\ \sqrt{x+1}}\ dx = ln\left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + k$$

e) $\int \frac{1}{x+\sqrt{x}}\ dx$

Canvi: $x = t^2 \quad \to \quad dx = 2t\ dt$

$$\int \frac{1}{x+\sqrt{x}}\ dx = \int \frac{2t\ dt}{t^2+t} = \int \frac{2\ dt}{t+1} = 2\ ln|t+1| + k =$$

$$= 2\ ln(\sqrt{x}+1) + k$$

f) $\int \frac{\sqrt{x}}{1+x}\ dx$

Canvi: $x = t^2 \quad \to \quad dx = 2t\ dt$

$$\int \frac{\sqrt{x}}{1+x}\ dx = \int \frac{t \cdot 2t\ dt}{1+t^2} = \int \frac{2t^2\ dt}{1+t^2} = \int \left(2 - \frac{2}{1+t^2} \right) dt =$$

$$= 2t - 2\ arc\ tg\ t + k = 2\sqrt{x} - 2\ arc\ tg\ \sqrt{x} + k$$

41. Resol, utilitzant un canvi de variable, aquestes integrals:

a) $\int \sqrt{1-x^2}\ dx$ **b)** $\int \frac{dx}{e^{2x}-3e^x}$ **c)** $\int \frac{e^{3x}-e^x}{e^{2x}+1} dx$ **d)** $\int \frac{1}{1+\sqrt{x}} dx$

☛ ***a) Fes*** $x = sin\ t$.

a) $x = sin\ t$

$dx = cos\ t\ dt$

$$\int \sqrt{1-sin^2\ t}\ \ cos\ t\ dt = \int cos\ t \cdot cos\ t\ dt = \int cos^2\ t\ dt = \frac{t + sin\ t \cdot cos\ t}{2} + k =$$

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$$= \frac{arc\ sin\ x}{2} + \frac{x \cdot \sqrt{1-x^2}}{2} + k$$

b) $\int \frac{dx}{e^{2x} - 3e^x} = \int \frac{1}{t^2 - 3t} \cdot \frac{1}{t}\, dt = \int \frac{1}{t^3 - 3t^2}\, dt = \int \left(\frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-3} \right) dt =$

$e^x = t \quad x = \ln t$
$e^x\, dx = dt$

$$= \int \frac{At(t-3) + B(t-3) + Ct^2}{t^3 - 3t^2}\, dt = \int \frac{At^2 - 3At + Bt - 3B + Ct^2}{t^3 - 3t^2}\, dt =$$

Feu sistema:

$$\begin{cases} A + C = 0 \\ -3A + B = 0 \\ -3B = 1 \end{cases} \qquad \begin{array}{l} A = -1/9 \\ B = -1/3 \\ C = 1/9 \end{array}$$

$$= \int \left(\frac{-1/9}{t} + \frac{-1/3}{t^2} + \frac{1/9}{t-3} \right) dt = \frac{-1}{9} \ln|t| + \frac{1}{3} t^{-1} + \frac{1}{9} \ln|t-3| + k =$$

desfeu el canvi

$$= \frac{-1}{9} \ln e^x + \frac{1}{3} (e^x)^{-1} + \frac{1}{9} \ln|e^x - 3| + k = \frac{-x}{9} + \frac{1}{3e^x} + \frac{1}{9} \ln|e^x - 3| + k$$

c) $\int \frac{e^{3x} - e^x}{e^{2x} + 1}\, dx$

$$\int \frac{t^3 - t}{t^2 + 1} \cdot \frac{dt}{t} = \int \frac{t^2 - 1}{t^2 + 1}\, dt = \int \left(1 + \frac{-2}{t^2 + 1} \right) dt = t - 2 \,\mathrm{arctg}\, t + k = e^x - 2 \,\mathrm{arctg}\, e^x + k$$

d) $\int \frac{1}{1 + \sqrt{x}}\, dx = \int \frac{1}{1 + t} \cdot 2t\, dt = \int \frac{2t}{1 + t}\, dt = 2\int \left(1 + \frac{-1}{t + 1} \right) dt =$

$\sqrt{x} = t$
$\frac{1}{2\sqrt{x}}\, dx = dt$

$$= 2[t - \ln|t + 1|] + k = 2\sqrt{x} - \ln|\sqrt{x} + 1| + k$$

42. **Troba la primitiva de $f(x) = \frac{1}{1 + 3x}$ que s'anul·la per a $x = 0$.**

$$F(x) = \int \frac{1}{1 + 3x}\, dx = \frac{1}{3} \int \frac{3}{1 + 3x}\, dx = \frac{1}{3} \ln|1 + 3x| + k$$

$$F(0) = k = 0$$

Per tant: $F(x) = \frac{-1}{3} \ln|1 + 3x|$

43. **Troba la funció F per a la qual $F'(x) = \dfrac{1}{x^2}$ i $F(1) = 2$.**

$$F(x) = \int \frac{1}{x^2}\, dx = \frac{-1}{x} + k$$

$$F(1) = -1 + k = 2 \quad \Rightarrow \quad k = 3$$

Per tant: $F(x) = \dfrac{-1}{x} + 3$

44. **De totes les primitives de la funció $y = 4x - 6$, quina d'aquestes pren el valor 4 per a $x = 1$?**

$$F(x) = \int (4x - 6)\, dx = 2x^2 - 6x + k$$

$$F(1) = 2 - 6 + k = 4 \quad \Rightarrow \quad k = 8$$

Per tant: $F(x) = 2x^2 - 6x + 8$

45. **Troba $f(x)$ si saps que $f''(x) = 6x$, $f'(0) = 1$ i $f(2) = 5$.**

$$\left.\begin{array}{l} f'(x) = \int 6x\, dx = 3x^2 + c \\ f'(0) = c = 1 \end{array}\right\} \quad f'(x) = 3x^2 + 1$$

$$\left.\begin{array}{l} f(x) = \int (3x^2 + 1)\, dx = x^3 + x + k \\ f(2) = 10 + k = 5 \quad \Rightarrow \quad k = -5 \end{array}\right\}$$

Per tant: $f(x) = x^3 + x - 5$

46. **Resol les integrals següents per substitució:**

a) $\displaystyle\int \frac{e^x}{1 - \sqrt{e^x}}\, dx$ **b)** $\displaystyle\int \sqrt{e^x - 1}\, dx$

☛ ***a) Fes $\sqrt{e^x} = t$. b) Fes $\sqrt{e^x - 1} = t$.***

a) $\displaystyle\int \frac{e^x}{1 - \sqrt{e^x}}\, dx$

Canvi: $\sqrt{e^x} = t \;\rightarrow\; e^{x/2} = t \;\rightarrow\; \dfrac{x}{2} = \ln t \;\rightarrow\; dx = \dfrac{2}{t}\, dt$

$$\int \frac{e^x}{1 - \sqrt{e^x}} = \int \frac{t^2 \cdot (2/t)\, dt}{1 - t} = \int \frac{2t\, dt}{1 - t} = \int \left(-2 + \frac{2}{1 - t}\right) dt =$$

$$= -2t - 2\ln|1 - t| + k = -2\sqrt{e^x} - 2\ln|1 - \sqrt{e^x}| + k$$

b) $\int \sqrt{e^x - 1}\, dx$

Canvi: $\sqrt{e^x - 1} = t \rightarrow e^x = t^2 + 1 \rightarrow x = ln\,(t^2 + 1) \rightarrow dx = \frac{2t}{t^2 + 1}\, dt$

$$\int \sqrt{e^x - 1}\, dx = \int t \cdot \frac{2t}{t^2 + 1}\, dt = \int \frac{2t^2}{t^2 + 1}\, dt = \int \left(2 - \frac{2}{t^2 + 1}\right) dt =$$

$$= 2t - 2\, arc\ tg\ t + k = 2\sqrt{e^x - 1} - 2\, arc\ tg \sqrt{e^x - 1} + k$$

47. **Calcula** $\int \frac{sin^2 x}{1 + cos\, x} dx$.

☛ ***Multiplica numerador i denominador per 1 – cos x.***

$$\int \frac{sin^2 x}{1 + cos\, x}\, dx = \int \frac{sin^2 x\,(1 - cos\, x)}{(1 + cos\, x)(1 - cos\, x)}\, dx = \int \frac{sin^2 x\,(1 - cos\, x)}{1 - cos^2 x}\, dx =$$

$$= \int \frac{sin^2 x\,(1 - cos\, x)}{sin^2 x}\, dx = \int (1 - cos\, x)\, dx = x - sin\, x + k$$

48. **Donada la funció f: $\mathbb{R} \rightarrow \mathbb{R}$ definida per $f(x) = ln\,(1 + x^2)$ troba la primitiva de f la gràfica de la qual passa per l'origen de la coordenada.**

$$\int ln\,(1 + x^2)\, dx = \int 1 \cdot ln\,(1 + x^2)\, dx = x \cdot ln\,(1 + x^2) - \int x \cdot \frac{2x}{1 + x^2}\, dx =$$

$u = ln\,(1 + x^2)$	$v = x$
$du = \frac{2x}{1 + x^2}$	$dv = 1$

$$= x \cdot ln\,(1 + x^2) - 2\int \frac{x^2}{1 + x^2}\, dx = x \cdot ln\,(1 + x^2) - 2\int \left(1 - \frac{1}{x^2 + 1}\right) dt =$$

$$= x \cdot ln\,(1 + x^2) - 2x + 2arctg\, x + k$$

Si ha de passar pel (0, 0)

$F(0) = 0$

$0 + k = 0 \rightarrow k = 0 \rightarrow F(x) = x \cdot ln\,(1 + x^2) - 2x + 2arctg\, x$

49. **D'una funció $y = f(x)$, $x > -1$ en sabem que té per derivada $y' = \frac{a}{1 + x}$ en què a és una constant. Determina la funció si, a més, sabem que $f(0) = 1$ i $f(1) = -1$.**

$$\int \frac{a}{1 + x}\, dx = a \cdot ln\, |1 + x| + k = f(x)$$

$$\left.\begin{array}{ll} f(0) = 1 & a \cdot ln\, 1 + k = 1 \\ f(1) = -1 & a \cdot ln\, 2 + k = -1 \end{array}\right\} \begin{array}{l} k = 1 \\ a = \frac{-2}{ln\, 2} \end{array}$$

$$f(x) = \frac{-2}{ln\, 2} \cdot ln\, |1 + x| + 1$$

50. **Calcula el valor de a per tal que una primitiva de $\int (ax^2 + x \cos x + 1)\, dx$ passi pel punt $(\pi, -1)$.**

$$\int (ax^2 + \cos x + 1)\, dx = \frac{a}{3} x^3 + \cos x + x \sin x + x + k = f(x)$$

$f(\pi) = -1$ $\qquad \frac{a}{3}\pi^3 + \cos \pi + \pi \sin \pi + \pi + k = -1$

$$\frac{a}{3}\pi^3 - 1 + \pi + k = -1$$

$$\frac{a}{3}\pi^3 + \pi + k = 0$$

De les opcions possibles una seria $k = 0 \rightarrow \frac{a}{3}\pi^3 + \pi = 0 \rightarrow a = \frac{-3}{\pi^2}$

51. **De la funció $f(x)$ se sap que $f'(x) = \frac{3}{(x+1)^2}$ i que $f(2) = 0$.**

a) Determina f.

b) Troba la primitiva de f la gràfica de la qual passa pel punt (0, 1).

a) $\int \frac{3}{(x+1)^2}\, dx = \frac{-3}{(x+1)} + k = f(x)$

Si $f(2) = 0$ $\qquad f(2) = \frac{-3}{2+1} + k = 0 \rightarrow k = 1 \rightarrow f(x) = \frac{-3}{(x+1)} + 1$

b) Si $f(0) = 1$ $\qquad f(0) = \frac{-3}{1} + k = 0 \rightarrow k = 3 \rightarrow f(x) = \frac{-3}{(x+1)} + 3$

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52. **Troba una primitiva de la funció:**

$$f(x) = x^2 \sin x$$

el valor de la qual per a $x = \pi$ sigui 4.

$$F(x) = \int x^2 \sin x\, dx$$

Integrem per parts:

$$\begin{cases} u = x^2 \rightarrow du = 2x\, dx \\ dv = \sin x\, dx \rightarrow v = -\cos x \end{cases}$$

$$F(x) = -x^2 \cos x + 2 \underbrace{\int x \cos x\, dx}_{I_1}$$

$$\begin{cases} u_1 = x & \rightarrow & du_1 = dx \\ dv_1 = \cos x\, dx & \rightarrow & v_1 = \sin x \end{cases}$$

$$I_1 = x \sin x - \int \sin x\, dx = x \sin x + \cos x$$

Per tant:

$$\left.\begin{array}{l} F(x) = -x^2 \cos x + 2\, x \sin x + 2 \cos x + k \\ F(\pi) = \pi^2 - 2 + k = 4 \quad \Rightarrow \quad k = 6 - \pi^2 \end{array}\right\}$$

$$F(x) = -x^2 \cos x + 2\, x \sin x + 2 \cos x + 6 - \pi^2$$

53. Determina la funció $f(x)$ si sabem que:

$$f''(x) = x \ln x,\ f'(1) = 0 \quad \text{i} \quad f(e) = \frac{e}{4}$$

$$f'(x) = \int x \ln x\, dx$$

Integrem per parts:

$$\begin{cases} u = \ln x & \rightarrow & du = \dfrac{1}{x}\, dx \\ dv = x\, dx & \rightarrow & v = \dfrac{x^2}{2} \end{cases}$$

$$\left.\begin{array}{l} f'(x) = \dfrac{x^2}{2} \ln x - \displaystyle\int \dfrac{x}{2}\, dx = \dfrac{x^2}{2} \ln x - \dfrac{x^2}{4} + k = \dfrac{x^2}{2}\left(\ln x - \dfrac{1}{2}\right) + k \\ f'(1) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right) + k = -\dfrac{1}{4} + k = 0 \quad \Rightarrow \quad k = \dfrac{1}{4} \end{array}\right\}$$

$$f'(x) = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + \frac{1}{4}$$

$$f(x) = \int \left[\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + \frac{1}{4}\right] dx = \underbrace{\int \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) dx}_{I} + \frac{1}{4}\, x$$

$$\begin{cases} u = \left(\ln x - \dfrac{1}{2}\right) & \rightarrow & du = \dfrac{1}{x}\, dx \\ dv = \dfrac{x^2}{2}\, dx & \rightarrow & v = \dfrac{x^3}{6} \end{cases}$$

$$I = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \int \frac{x^2}{6}\, dx = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \frac{x^3}{18} + k$$

Per tant:

$$\left.\begin{array}{l} f(x) = \dfrac{x^3}{6}\left(ln\ x - \dfrac{1}{2}\right) - \dfrac{x^3}{18} + \dfrac{1}{4}x + k \\ f(e) = \dfrac{e^3}{12} - \dfrac{e^3}{18} + \dfrac{e}{4} + k = \dfrac{e^3}{36} + \dfrac{e}{4} + k = \dfrac{e}{4} \quad \Rightarrow \quad k = -\dfrac{e^3}{36} \end{array}\right\}$$

$$f(x) = \frac{x^3}{6}\left(ln\ x - \frac{1}{2}\right) - \frac{x^3}{18} + \frac{1}{4}x - \frac{e^3}{36}$$

54. **Calcula l'expressió d'una funció $f(x)$ tal que $f'(x) = x\, e^{-x^2}$ i que $f(0) = \frac{1}{2}$.**

$$f(x) = \int x\, e^{-x^2}\, dx = -\frac{1}{2}\int -2x\, e^{-x^2}\, dx = -\frac{1}{2}\, e^{-x^2} + k$$

$$f(0) = -\frac{1}{2} + k = \frac{1}{2} \quad \Rightarrow \quad k = 1$$

Per tant: $f(x) = -\frac{1}{2}\, e^{-x^2} + 1$

55. **Troba la funció derivable $f: [-1, 1] \to \mathbb{R}$ que compleix $f(1) = -1$ i**

$$f'(x) = \begin{cases} x^2 - 2x & \text{si } -1 \le x < 0 \\ e^x - 1 & \text{si } 0 \le x \le 1 \end{cases}$$

- Si $x \neq 0$:

$$f(x) = \begin{cases} \dfrac{x^3}{3} - x^2 + k & \text{si } -1 \le x < 0 \\ e^x - x + c & \text{si } 0 < x \le 1 \end{cases}$$

- Busquem k i c tenint en compte que $f(1) = -1$ i que $f(x)$ ha de ser contínua en $x = 0$.

$$f(1) = -1 \quad \Rightarrow \quad e - 1 + c = -1 \quad \Rightarrow \quad c = -e$$

$$\left.\begin{array}{l} \lim\limits_{x \to 0^-} f(x) = k \\ \lim\limits_{x \to 0^+} f(x) = 1 - e \end{array}\right\} \; k = 1 - e$$

Per tant: $f(x) = \begin{cases} \dfrac{x^3}{3} - x^2 + 1 - e & \text{si } -1 \le x < 0 \\ e^x - x - e & \text{si } 0 \le x \le 1 \end{cases}$

56. **D'una funció derivable se sap que passa pel punt A(–1, –4) i que la derivada és:**

$$f'(x) = \begin{cases} 2 - x & \text{si } x \le 1 \\ 1/x & \text{si } x > 1 \end{cases}$$

a) Troba l'expressió de $f(x)$.

b) Obtén l'equació de la recta tangent a $f(x)$ en $x = 2$.

a) Si $x \neq 1$:

$$f(x) = \begin{cases} 2x - \dfrac{x^2}{2} + k & \text{si } x < 1 \\ \ln x + c & \text{si } x > 1 \end{cases}$$

Busquem k i c tenint en compte que $f(-1) = -4$ i que $f(x)$ ha de ser contínua en $x = 1$.

$$f(-1) = -\frac{5}{2} + k = -4 \quad \Rightarrow \quad k = -\frac{3}{2}$$

$$\left.\begin{aligned} &\lim_{x \to 1^-} f(x) = \frac{3}{2} - \frac{3}{2} = 0 \\ &\lim_{x \to 1^+} f(x) = c \end{aligned}\right\} \quad c = 0$$

Per tant: $f(x) = \begin{cases} 2x - \dfrac{x^2}{2} - \dfrac{3}{2} & \text{si } x < 1 \\ \ln x & \text{si } x \ge 1 \end{cases}$

b) $f(2) = \ln 2$; $f'(2) = \dfrac{1}{2}$

L'equació de la recta tangent serà: $y = \ln 2 + \dfrac{1}{2}(x - 2)$.

57. **Calcula:**

a) $\displaystyle\int |1 - x|\, dx$ **b)** $\displaystyle\int (3 + |x|)\, dx$ **c)** $\displaystyle\int |2x - 1|\, dx$ **d)** $\displaystyle\int \left|\frac{x}{2} - 2\right| dx$

a) $\displaystyle\int |1 - x|\, dx$

$$|1 - x| = \begin{cases} 1 - x & \text{si } x < 1 \\ -1 + x & \text{si } x \ge 1 \end{cases}$$

$$f(x) = \int |1 - x|\, dx = \begin{cases} x - \dfrac{x^2}{2} + k & \text{si } x < 1 \\ -x + \dfrac{x^2}{2} + c & \text{si } x \ge 1 \end{cases}$$

En $x = 1$, la funció ha de ser contínua:

$$\left.\begin{aligned} &\lim_{x \to 1^-} f(x) = \frac{1}{2} + k \\ &\lim_{x \to 1^+} f(x) = -\frac{1}{2} + c \end{aligned}\right\} \quad \frac{1}{2} + k = -\frac{1}{2} + c \quad \Rightarrow \quad c = 1 + k$$

Per tant:

$$\int |1 - x|\, dx = \begin{cases} x - \dfrac{x^2}{2} + k & \text{si } x < 1 \\ -x + \dfrac{x^2}{2} + 1 + k & \text{si } x \geq 1 \end{cases}$$

b) $\int (3 + |x|)\, dx$

$$3 + |x| = \begin{cases} 3 - x & \text{si } x < 0 \\ 3 + x & \text{si } x \geq 0 \end{cases}$$

$$f(x) = \int (3 + |x|)\, dx = \begin{cases} 3x - \dfrac{x^2}{2} + k & \text{si } x < 0 \\ 3x + \dfrac{x^2}{2} + c & \text{si } x \geq 0 \end{cases}$$

En $x = 0$, $f(x)$ ha de ser contínua:

$$\left.\begin{aligned} &\lim_{x \to 0^-} f(x) = k \\ &\lim_{x \to 0^+} f(x) = c \end{aligned}\right\} \quad c = k$$

Per tant:

$$\int (3 + |x|)\, dx = \begin{cases} 3x - \dfrac{x^2}{2} + k & \text{si } x < 0 \\ 3x + \dfrac{x^2}{2} + k & \text{si } x \geq 0 \end{cases}$$

c) $\int |2x - 1|\, dx$

$$|2x - 1| = \begin{cases} -2x + 1 & \text{si } x < 1/2 \\ 2x - 1 & \text{si } x \geq 1/2 \end{cases}$$

$$f(x) = \int |2x - 1|\, dx = \begin{cases} -x^2 + x + k & \text{si } x < \dfrac{1}{2} \\ x^2 - x + c & \text{si } x \geq \dfrac{1}{2} \end{cases}$$

$f(x)$ ha de ser contínua en $x = \frac{1}{2}$:

$$\left.\begin{array}{l} \lim\limits_{x \to (1/2)^-} f(x) = \frac{1}{4} + k \\ \lim\limits_{x \to (1/2)^+} f(x) = -\frac{1}{4} + c \end{array}\right\} \frac{1}{4} + k = -\frac{1}{4} + c \quad \Rightarrow \quad c = \frac{1}{2} + k$$

Per tant:

$$\int |2x - 1|\, dx = \begin{cases} -x^2 + x + k & \text{si } x < \frac{1}{2} \\ x^2 - x + \frac{1}{2} + k & \text{si } x \geq \frac{1}{2} \end{cases}$$

d) $\int \left| \frac{x}{2} - 2 \right| dx$

$$\left| \frac{x}{2} - 2 \right| = \begin{cases} -\frac{x}{2} + 2 & \text{si } x < 4 \\ \frac{x}{2} - 2 & \text{si } x \geq 4 \end{cases}$$

$$f(x) = \int \left| \frac{x}{2} - 2 \right| dx = \begin{cases} -\frac{x^2}{4} + 2x + k & \text{si } x < 4 \\ \frac{x^2}{4} - 2x + c & \text{si } x \geq 4 \end{cases}$$

$f(x)$ ha de ser contínua en $x = 4$:

$$\left.\begin{array}{l} \lim\limits_{x \to 4^-} f(x) = 4 + k \\ \lim\limits_{x \to 4^+} f(x) = -4 + c \end{array}\right\} 4 + k = -4 + c \quad \Rightarrow \quad c = 8 + k$$

Per tant:

$$\int \left| \frac{x}{2} - 2 \right| dx = \begin{cases} -\frac{x^2}{4} + 2x + k & \text{si } x < 4 \\ \frac{x^2}{4} - 2x + 8 + k & \text{si } x \geq 4 \end{cases}$$

58. Calcula $\int \frac{1}{\sin^2 x \cos^2 x} dx$.

$$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx =$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx =$$

$$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx = tg\, x - cotg\, x + k$$

59. **Troba la primitiva $F(x)$ de la funció $f(x) = 2x$ tal que $F(x) \leq 0$ en l'interval [–2, 2].**

$\int 2x\, dx = x^2 + k = F(x)$

Si $F(x) \leq 0$ en interval [–2, 2] vol dir que la paràbola talla l'eix horitzontal en $x = 2$ i $x = -2$.

Si $x = \pm 2$ ha de ser arrel de $F(x) \rightarrow (\pm 2)^2 + k = 0 \rightarrow k = -4 \rightarrow F(x) = x^2 - 4$

QÜESTIONS TEÒRIQUES

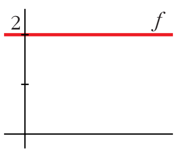
60. **Prova que, si $F(x)$ és una primitiva de $f(x)$ i C un nombre real qualsevol, la funció $F(x) + C$ és també una primitiva de $f(x)$.**

$F(x)$ primitiva de $f(x) \Leftrightarrow F'(x) = f(x)$

$\left(F(x) + C\right)' = F'(x) = f(x) \Rightarrow F(x) + C$ és primitiva de $f(x)$.

61. **Representa en cada cas tres primitives de la funció f la gràfica de la qual és la següent:**

a)



$f(x) = 2 \Rightarrow F(x) = 2x + k$

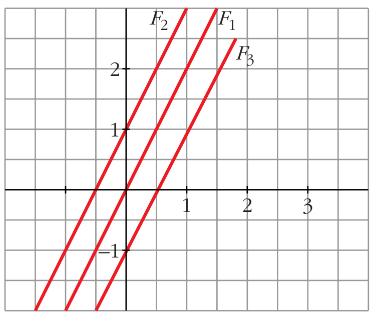
Per exemple:

$F_1(x) = 2x$

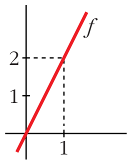
$F_2(x) = 2x + 1$

$F_3(x) = 2x - 1$

les gràfiques de les quals són:



b)



$f(x) = 2x \;\Rightarrow\; F(x) = x^2 + k$

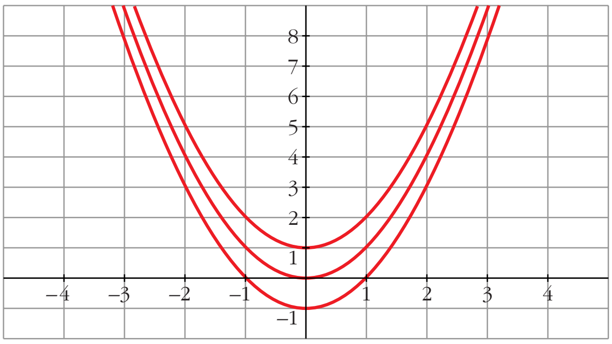
Per exemple:

$F_1(x) = x^2$

$F_2(x) = x^2 + 1$

$F_3(x) = x^2 - 1$

les gràfiques de les quals són:



62. **Saps que una primitiva de la funció $f(x) = \dfrac{1}{x}$ és $F(x) = ln\,|x|$. Per què es pren el valor absolut de x?**

$f(x) = \dfrac{1}{x}$ està definida per a tot $x \neq 0$; i és la derivada de la funció:

$$F(x) = \begin{cases} ln\,x & \text{si } x > 0 \\ ln\,(-x) & \text{si } x < 0 \end{cases}$$

és a dir, de $F(x) = ln|x|$.

63. **En una integral fem el canvi de variable $e^x = t$. Quina és l'expressió de dx en funció de t?**

$$e^x = t \;\rightarrow\; x = ln\,t \;\rightarrow\; dx = \frac{1}{t}\,dt$$

64. **Comprova que: $\displaystyle\int \frac{1}{cos\,x}\,dx = ln\,|sec\,x + tg\,x| + k$**

Hem de provar que la derivada de $f(x) = ln|sec\,x + tg\,x| + k$ és $f'(x) = \dfrac{1}{cos\,x}$.

Derivem $f(x) = ln\left|\dfrac{1 + sin\,x}{cos\,x}\right| + k$:

$$f'(x) = \frac{\dfrac{cos^2 x + sin\,x(1 + sin\,x)}{cos^2 x}}{\dfrac{1 + sin\,x}{cos\,x}} = \frac{\dfrac{cos^2 x + sin\,x + sin^2 x}{cos\,x}}{1 + sin\,x} =$$

$$= \frac{1 + sin\,x}{(1 + sin\,x)\;cos\,x} = \frac{1}{cos\,x}$$

65. Comprova que: $\displaystyle\int \frac{1}{sin\,x\;cos\,x}\,dx = ln\,|tg\,x| + k$

Hem de comprovar que la derivada de la funció $f(x) = ln|\,tg\,x| + k$ és $f'(x) = \dfrac{1}{sin\,x\,cos\,x}$.

Derivem $f(x)$:

$$f'(x) = \frac{1/cos^2 x}{tg\,x} = \frac{1/cos^2 x}{sin\,x/cos\,x} = \frac{1}{sin\,x\,cos\,x}$$

66. Sense utilitzar-hi càlcul de derivades, prova que:

$$F(x) = \frac{1}{1 + x^4} \quad \text{i} \quad G(x) = \frac{-x^4}{1 + x^4}$$

són dues primitives d'una mateixa funció.

Si $F(x)$ i $G(x)$ són dues primitives d'una mateixa funció, la seva diferència és una constant. Vegem-ho:

$$F(x) - G(x) = \frac{1}{1 + x^4} - \left(\frac{-x^4}{1 + x^4}\right) = \frac{1 + x^4}{1 + x^4} = 1$$

Per tant, hem obtingut que: $F(x) = G(x) + 1$

Així doncs, les dues són primitives d'una mateixa funció.

67. Siguin f i g dues funcions primitives i derivables que es diferencien en una constant. Podem assegurar que f i g tenen una mateixa primitiva?

No. Per exemple:

$$\left.\begin{array}{l} f(x) = 2x + 1 \;\rightarrow\; F(x) = x^2 + x + k \\ g(x) = 2x + 2 \;\rightarrow\; G(x) = x^2 + 2x + c \end{array}\right\}$$

$f(x)$ i $g(x)$ són contínues, derivables i es diferencien en una constant (ja que $f(x) = g(x) - 1$).

Tanmateix, les seves primitives, $F(x)$ i $G(x)$ respectivament, són diferents, qualssevol que siguin els valors de k i c.

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PER APROFUNDIR

68. **Per integrar una funció el denominador de la qual és un polinomi de segon grau sense arrels reals, distingirem dos casos:**

a) Si el numerador és constant, transformem el denominador per obtenir un binomi al quadrat. La solució serà un arc tangent:

$$\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x+2)^2 + 1}$$

(Completa'n la resolució.)

b) Si el numerador és de primer grau, es descompon en un logaritme neperià i un arc tangent:

$$\int \frac{(x+5)dx}{x^2 + 2x + 3} = \frac{1}{2}\int \frac{2x+10}{x^2+2x+3}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}dx + \frac{1}{2}\int \frac{8\,dx}{x^2+2x+3}$$

(Completa'n la resolució.)

a) $\displaystyle\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x+2)^2 + 1} = arc\ tg(x+2) + k$

b) $\displaystyle\int \frac{(x+5)\,dx}{x^2 + 2x + 3} = \frac{1}{2}\int \frac{2x+10}{x^2+2x+3}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}dx + \frac{1}{2}\int \frac{8\ dx}{x^2+2x+3} =$

$$= \frac{1}{2}\ln(x^2+2x+3) + 4\int \frac{dx}{(x+1)^2+2} =$$

$$= \frac{1}{2}\ln(x^2+2x+3) + 2\int \frac{dx}{\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1} =$$

$$= \frac{1}{2}\ln(x^2+2x+3) + 2\sqrt{2}\int \frac{\left(1/\sqrt{2}\right)\,dx}{\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1} =$$

$$= \frac{1}{2}\ln(x^2+2x+3) + 2\sqrt{2}\ arc\ tg\left(\frac{x+1}{\sqrt{2}}\right) + k$$

69. **Observa com es resol aquesta integral:**

$$I = \int \frac{x+1}{x^3 + 2x^2 + 3x}dx$$

$$x^3 + 2x^2 + 3x = x(x^2 + 2x + 3)$$

La fracció es descompon així: $\displaystyle\frac{x+1}{x^3+2x^2+3x} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+3}$

Obtenim: $A = \frac{1}{3},\ B = -\frac{1}{3},\ C = \frac{1}{3}$

Substituïm: $I = \dfrac{1}{3}\displaystyle\int \dfrac{1}{x}dx - \dfrac{1}{3}\int \dfrac{x-1}{x^2+2x+3}dx$

(Completa'n la resolució.)

Completem la resolució:

$$I = \frac{1}{3}\int \frac{1}{x}\,dx - \frac{1}{3}\int \frac{x-1}{x^2+2x+3}\,dx =$$

$$= \frac{1}{3}\ln|x| - \frac{1}{6}\int \frac{2x-2}{x^2+2x+3}\,dx = \frac{1}{3}\ln|x| - \frac{1}{6}\int \frac{2x+2-4}{x^2+2x+3}\,dx =$$

$$= \frac{1}{3}\ln|x| - \frac{1}{6}\int \frac{2x-2}{x^2+2x+3}\,dx + \frac{2}{3}\int \frac{dx}{x^2+2x+3} \overset{(*)}{=}$$

$$= \frac{1}{3}\ln|x| - \frac{1}{6}\ln(x^2+2x+3) + \frac{\sqrt{2}}{3}\,arc\,tg\left(\frac{x+1}{\sqrt{2}}\right) + k$$

(*) $\left(\text{Vegeu en l'exercici 49, apartat b), el càlcul de } \displaystyle\int \frac{dx}{x^2+2x+3}\right)$.

70. **Resol les integrals següents:**

a) $\displaystyle\int \frac{2x-1}{x^3+x}dx$ **b)** $\displaystyle\int \frac{1}{x^3+1}dx$ **c)** $\displaystyle\int \frac{x^2+3x+8}{x^2+9}dx$

d) $\displaystyle\int \frac{2x+10}{x^2+x+1}dx$ **e)** $\displaystyle\int \frac{2}{x^2+3x+4}dx$ **f)** $\displaystyle\int \frac{dx}{(x+1)^2(x^2+1)}$

☛ ***e) Multiplica numerador i denominador per 4.***

a) $\displaystyle\int \frac{2x-1}{x^3+x}\,dx = \int \frac{2x-1}{x(x^2+1)}\,dx$

Descomponem la fracció:

$$\frac{2x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+Bx^2+Cx}{x(x^2+1)}$$

$$2x-1 = A(x^2+1) + Bx^2 + Cx$$

Busquem A, B i C:

$$\left.\begin{array}{lll} x=0 & \rightarrow & -1 = A \\ x=1 & \rightarrow & 1 = 2A+B+C \;\rightarrow\; 3 = B+C \\ x=-1 & \rightarrow & -3 = 2A+B-C \;\rightarrow\; -1 = B-C \end{array}\right\} \begin{array}{l} A=-1 \\ B=1 \\ C=2 \end{array}$$

Per tant:

$$\int \frac{2x-1}{x^3+x}\,dx = \int \left(\frac{-1}{x} + \frac{x+2}{x^2+1}\right)dx =$$

$$= \int \frac{-1}{x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + 2\int \frac{dx}{x^2+1} =$$

$$= -\ln|x| + \frac{1}{2}\ln(x^2+1) + 2\,arc\,tg\,x + k$$

b) $\int \frac{1}{x^3+1}\,dx = \int \frac{dx}{(x+1)(x^2-x+1)}$

Descomponem la fracció:

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} =$$

$$= \frac{A(x^2-x+1) + Bx(x+1) + C(x+1)}{(x+1)(x^2-x+1)}$$

$$1 = A(x^2-x+1) + Bx(x+1) + C(x+1)$$

Busquem A, B i C:

$$\left.\begin{array}{llll} x=-1 & \rightarrow\ 1 = 3A & \rightarrow & A = 1/3 \\ x = 0 & \rightarrow\ 1 = A + C & \rightarrow & C = 2/3 \\ x = 1 & \rightarrow\ 1 = A + 2B + 2C & \rightarrow & B = -1/3 \end{array}\right\}$$

Per tant:

$$\int \frac{1}{x^3+1}\,dx = \int \frac{-1/3}{x+1}\,dx + \int \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\int \frac{2x-4}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\int \frac{2x-1-3}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2-x+1} =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\,ln(x^2-x+1) + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\,ln(x^2-x+1) + \frac{1}{2}\int \frac{4/3}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\,ln(x^2-x+1) + \frac{\sqrt{3}}{3}\int \frac{2/\sqrt{3}}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1}\,dx =$$

$$= \frac{1}{3}\,ln|x+1| - \frac{1}{6}\,ln(x^2-x+1) + \frac{\sqrt{3}}{3}\,arc\ tg\left(\frac{2x-1}{\sqrt{3}}\right) + k$$

c) $\displaystyle\int \frac{x^2 + 3x + 8}{x^2 + 9}\, dx = \int \left(1 + \frac{3x - 1}{x^2 + 9}\right) dx = x + \int \frac{3x}{x^2 + 9}\, dx - \int \frac{dx}{x^2 + 9} =$

$\displaystyle = x + \frac{3}{2}\int \frac{2x}{x^2 + 9}\, dx - \int \frac{1/9}{(x/3)^2 + 1}\, dx =$

$\displaystyle = x + \frac{3}{2}\, ln(x^2 + 9) - \frac{1}{3}\, arc\ tg\left(\frac{x}{3}\right) + k$

d) $\displaystyle\int \frac{2x + 10}{x^2 + x + 1}\, dx = \int \frac{2x + 1 + 9}{x^2 + x + 1}\, dx = \int \frac{2x + 1}{x^2 + x + 1}\, dx + 9\int \frac{1}{x^2 + x + 1}\, dx =$

$\displaystyle = ln(x^2 + x + 1) + 9\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} =$

$\displaystyle = ln(x^2 + x + 1) + 6\sqrt{3}\int \frac{2/\sqrt{3}}{\left(\frac{2x + 1}{\sqrt{3}}\right)^2 + 1}\, dx =$

$\displaystyle = ln(x^2 + x + 1) + 6\sqrt{3}\ arc\ tg\left(\frac{2x + 1}{\sqrt{3}}\right) + k$

e) $\displaystyle\int \frac{2}{x^2 + 3x + 4}\, dx = \int \frac{8}{4x^2 + 12x + 16}\, dx = \int \frac{8}{(2x + 3)^2 + 7}\, dx =$

$\displaystyle = \int \frac{8/7}{\left(\frac{2x + 3}{\sqrt{7}}\right)^2 + 1}\, dx = \frac{8}{7} \cdot \frac{\sqrt{7}}{2} \int \frac{2/\sqrt{7}}{\left(\frac{2x + 3}{\sqrt{7}}\right)^2 + 1}\, dx =$

$\displaystyle = \frac{4\sqrt{7}}{7}\ arc\ tg\left(\frac{2x + 3}{\sqrt{7}}\right) + k$

f) $\displaystyle\int \frac{dx}{(x + 1)^2\,(x^2 + 1)}$

Descomponem la fracció:

$$\frac{1}{(x + 1)^2\,(x^2 + 1)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{Cx + D}{x^2 + 1}$$

$$1 = A(x + 1)(x^2 + 1) + B(x^2 + 1) + Cx(x + 1)^2 + D(x + 1)^2$$

Trobem A, B, C i D:

$$\left.\begin{array}{lll} x = -1 & \rightarrow\ 1 = 2B & \rightarrow\ B = 1/2 \\ x = 0 & \rightarrow\ 1 = A + B + D & \\ x = 1 & \rightarrow\ 1 = 4A + 2B + 4C + 4D & \\ x = -2 & \rightarrow\ 1 = -5A + 5B - 2C + D & \end{array}\right\} \begin{array}{l} A = 1/2 \\ B = 1/2 \\ C = -1/2 \\ D = 0 \end{array}$$

Per tant:

$$\int \frac{dx}{(x+1)^2(x^2+1)} = \int \left(\frac{1/2}{x+1} + \frac{1/2}{(x+1)^2} - \frac{1}{2} \cdot \frac{x}{x^2+1} \right) dx =$$

$$= \frac{1}{2} \ln|x+1| - \frac{1}{2(x+1)} - \frac{1}{4} \ln(x^2+1) + k$$

PER PENSAR UNA MICA MÉS

71. **S'anomena equació diferencial de primer ordre una equació en què, a més de les variables x i y, figura també y'. Resoldre una equació diferencial és buscar una funció $y = f(x)$ que verifiqui l'equació proposada.**

Per exemple, l'equació $x\,y^2 + y' = 0$ es resol així:

$$y' = -x\,y^2 \;\rightarrow\; \frac{dy}{dx} = -x\,y^2 \;\rightarrow\; dy = -x\,y^2\,dx$$

En separem les variables:

$$\frac{dy}{y^2} = -x\,dx \;\rightarrow\; \int \frac{dy}{y^2} = \int (-x)\,dx$$

$$-\frac{1}{y} = -\frac{x^2}{2} + k \;\rightarrow\; y = \frac{2}{x^2 - 2k}$$

Hi ha infinites solucions.

Busca la solució que passa pel punt (0, 2) i comprova que la corba que obtens verifica l'equació proposada.

- Busquem la solució que passa pel punt $(0, 2)$:

$$y = \frac{2}{x^2 - 2k} \;\rightarrow\; 2 = \frac{2}{-2k} \;\Rightarrow\; -4k = 2 \;\Rightarrow\; k = \frac{-1}{2}$$

Per tant: $y = \dfrac{2}{x^2+1}$

- Comprovem que verifica l'equació $xy^2 + y' = 0$:

$$xy^2 + y' = x\left(\frac{2}{x^2+1}\right)^2 - \frac{4x}{(x^2+1)^2} = x \cdot \frac{4}{(x^2+1)^2} - \frac{4x}{(x^2+1)^2} =$$

$$= \frac{4x}{(x^2+1)^2} - \frac{4x}{(x^2+1)^2} = 0$$

72. **Resol les equacions següents:**

a) $y\,y' - x = 0$ **b)** $y^2\,y' - x^2 = 1$ **c)** $y' - x\,y = 0$

d) $y'\sqrt{x} - y = 0$ **e)** $y'\,e^y + 1 = e^x$ **f)** $x^2\,y' + y^2 + 1 = 0$

a) $y\,y' - x = 0$

$$y' = \frac{x}{y} \;\Rightarrow\; \frac{dy}{dx} = \frac{x}{y} \;\Rightarrow\; y\,dy = x\,dx \;\Rightarrow\; \int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + k \;\Rightarrow\; y^2 = x^2 + 2k$$

b) $y^2\,y' - x^2 = 1$

$$y' = \frac{1 + x^2}{y^2} \;\Rightarrow\; \frac{dy}{dx} = \frac{1 + x^2}{y^2} \;\Rightarrow\; y^2\,dy = (1 + x^2)\,dx$$

$$\int y^2\,dy = \int (1 + x^2)\,dx \;\Rightarrow\; \frac{y^3}{3} = x + \frac{x^3}{3} + k \;\Rightarrow$$

$$\Rightarrow\; y^3 = 3x + x^3 + 3k \;\Rightarrow\; y = \sqrt[3]{3x + x^3 + 3k}$$

c) $y' - x\,y = 0$

$$y' = xy \;\Rightarrow\; \frac{dy}{dx} = xy \;\Rightarrow\; \frac{dy}{y} = x\,dx \;\Rightarrow\; \int \frac{dy}{y} = \int x\,dx$$

$$ln\,|y| = \frac{x^2}{2} + k \;\Rightarrow\; |y| = e^{(x^2/2) + k}$$

d) $y'\sqrt{x} - y = 0$

$$y' = \frac{y}{\sqrt{x}} \;\Rightarrow\; \frac{dy}{dx} = \frac{y}{\sqrt{x}} \;\Rightarrow\; \frac{dy}{y} = \frac{dx}{\sqrt{x}} \;\Rightarrow\; \int \frac{dy}{y} = \int \frac{dx}{\sqrt{x}}$$

$$ln\,|y| = 2\sqrt{x} + k \;\Rightarrow\; |y| = e^{2\sqrt{x} + k}$$

e) $y'\,e^y + 1 = e^x$

$$y' = \frac{e^x - 1}{e^y} \;\Rightarrow\; \frac{dy}{dx} = \frac{e^x - 1}{e^y}$$

$$e^y\,dy = (e^x - 1)\,dx \;\Rightarrow\; \int e^y\,dy = \int (e^x - 1)\,dx$$

$$e^y = e^x - x + k \;\Rightarrow\; y = ln(e^x - x + k)$$

f) $x^2\,y' + y^2 + 1 = 0$

$$y' = \frac{-1 - y^2}{x^2} \;\Rightarrow\; \frac{dy}{dx} = \frac{-(1 + y^2)}{x^2} \;\Rightarrow\; \frac{dy}{1 + y^2} = \frac{-1}{x^2}\,dx$$

$$\int \frac{dy}{1 + y^2} = \int \frac{-1}{x^2}\,dx \;\Rightarrow\; arc\,tg\,y = \frac{1}{x} + k$$

$$y = tg\left(\frac{1}{x} + k\right)$$